Cocycles, symplectic structures and intersection

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1. Introduction

In the paper [B-C-G] Besson, Courtois and Gallot proved the following remarkable theorem: Let S be a closed rank 1 locally symmetric space of noncompact type and let M be a closed manifold of negative curvature which is homotopy equivalent to S. If S and M have the same volume and the same volume entropies (i.e. the same asymptotic growth rate of volumes of balls in their universal covers), then S and M are isometric.

One application of this result is the solution of the so-called *conjugacy problem* for locally symmetric manifolds. This problem can be stated in the following way: The *marked length spectrum* of a closed negatively curved manifold (M, g) is the function ρ_M which assigns to each conjugacy class in the fundamental group $\pi_1(M)$ of M the length of the unique g-geodesic representing the class. Two homotopy equivalent negatively curved manifolds M, N have *the same marked length spectrum* if there is an isomorphism $\Psi : \pi_1(M) \to \pi_1(N)$ such that $\rho_N \circ \Psi = \rho_M$. This is equivalent to the existence of a continuous *time preserving conjugacy* for the *geodesic flows* for M and N ([H1]). Such a conjugacy is defined to be a homeomorphism Λ of the unit tangent bundle T^1M of M onto the unit tangent bundle T^1N of N which is equivariant under the action of the geodesic flows Φ^t on T^1M and T^1N ([H1]). The conjugacy problem now asks whether a composition of Λ with the time-t-map of the geodesic flow on T^1N for a suitable $t \in \mathbf{R}$ in the restriction to T^1M of the differential of an isometry of M onto N. This is known to be true for surfaces ([O]).

Since the volume entropy of a closed negatively curved manifold equals the topological entropy of the geodesic flow, two manifolds with time preserving conjugate geodesic flows have the same volume entropy.

Let X^0 be the *geodesic spray*, i.e. the generator of the geodesic flow Φ^t. Recall that there is a Hölder continuous Φ^t-invariant decomposition $TT^1M = \mathbf{R}X^0 \oplus TW^{ss} \oplus TW^{su}$, the so called *Anosov splitting*, where TW^{ss} (or TW^{su}) is the tangent bundle of the *strong stable (or strong unstable) foliation* W^{ss} (or W^{su}) on T^1M. The bundle $TW^{ss} \oplus TW^{su}$ is smooth, and hence we obtain a smooth 1-form ω on T^1M by defining $\omega(X^0) \equiv 1$ and $\omega(TW^{ss} \oplus TW^{su}) = 0$. This form ω is called the *canonical contact form*. The differential form $\omega \wedge (d\omega)^{n-1}$ is the volume form of the so called *Sasaki metric* on T^1M, and the total mass of T^1M with respect to this volume form equals the product of the volume of M with the volume of the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$.

The pull-back of the canonical contact form on T^1N under a time preserving conjugacy $\Lambda : T^1M \to T^1N$ of class C^1 is the canonical contact form on T^1M. This implies that M and N have the same volume if their geodesic flows are C^1-time-preserving conjugate. Therefore the result of Besson, Courtois and Gallot gives a positive answer to the

conjugacy problem if one of the manifolds under consideration is locally symmetric and if the conjugacy is assumed to be of class C^1.

One of the objectives of this note is to remove this regularity assumption on the conjugacy. In Section 3 we show:

Theorem A: *If M and N are closed negatively curved manifolds with the same marked length spectrum and if the Anosov splitting of TT^1M is of class C^1, then M and N have the same volume.*

Manifolds with strictly 1/4- pinched sectional curvature or locally symmetric manifolds have C^1 Anosov splitting. Thus as a corollary from [B-C-G] we obtain:

Corollary: *If M has the same marked length spectrum as a closed negatively curved locally symmetric space S, then M and S are isometric.*

Recall that the *space of geodesics* $\mathcal{G}\tilde{M}$ of the universal cover $\tilde{M}$ of (M, g) is the quotient of the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ under the action of the geodesic flow Φ^t. Also $\mathcal{G}\tilde{M}$ can naturally be identified with $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ where $\partial\tilde{M}$ is the *ideal boundary* of $\tilde{M}$ and Δ is the diagonal. The space of geodesics is a smooth manifold, and the differential $d\omega$ of the canonical contact form projects to a smooth symplectic structure on $\mathcal{G}\tilde{M}$. The fundamental group $\pi_1(M)$ of M acts on $\mathcal{G}\tilde{M}$ as a group of symplectomorphisms.

The *cross ratio* of the metric g is a function on $\mathcal{G}\tilde{M} \times \mathcal{G}\tilde{M}$. We show in Section 3 that this function can be viewed as a coarse (or rather integrated) version of the symplectic form $d\omega$ on $\mathcal{G}\tilde{M}$. This enables us to construct the measure λ_g on $\mathcal{G}\tilde{M}$ defined by the smooth volume form $(d\omega)^{n-1}$ in a purely combinatorial way from the cross ratio. The measure λ_g is usually called the *Lebesgue Liouville current* of the metric g. Our construction is general, but unfortunately so far we are only able to show that the measure we obtain from it is exactly the Lebesgue Liouville current (rather than some multiple of it) under the additional assumption that the Anosov splitting of TT^1M is of class C^1.

A *geodesic current* for M is a locally finite Borel measure on the space of geodesics which is invariant under the action of the fundamental group and under the exchange of the factors in the product decomposition of $\mathcal{G}\tilde{M} = \partial\tilde{M} \times \partial\tilde{M} - \Delta$. If M and N are two homotopy equivalent closed negatively curved manifolds, then there is a natural $\pi_1(M) = \pi_1(N)$- equivariant Hölder homeomorphism of $\partial\tilde{M}$ onto $\partial\tilde{N}$ which induces an equivariant homoeomorphism of $\mathcal{G}\tilde{M}$ onto $\mathcal{G}\tilde{N}$, in particular the spaces of geodesic currents for M and N are naturally identified.

With this identification we show in Section 4:

Theorem B: *Let (M, g), (N, h) be homotopy equivalent closed negatively curved manifolds. If $\lambda_g = \lambda_h$ and if the Anosov splittings of TT^1M and TT^1N are of class C^1, then M and N have the same marked length spectrum.*

Recall that a Hölder continuous *additive cocycle* for the geodesic flow Φ^t is a Hölder continuous function $\zeta: T^1M \times \mathbf{R} \to \mathbf{R}$ which satisfies $\zeta(v, s+t) = \zeta(v, s) + \zeta(\Phi^s v, t)$ for all $v \in T^1M$ and all $s, t \in \mathbf{R}$. Two cocycles ζ, ξ are called *cohomologous* if there is a Hölder

continuous function β on T^1M such that $\zeta(v,t)-\xi(v,t)=\beta(\Phi^t v)-\beta(v)$ for all $v\in T^1M$ and all $t\in\mathbf{R}$. We denote by $[\zeta]$ the cohomology class of ζ. Let $\mathcal{F}: T^1M\to T^1M$ be the *flip* $v\to\mathcal{F}v=-v$; the flip $\mathcal{F}\zeta$ of a cocycle ζ is the cocycle $\mathcal{F}\zeta(v,t)=\zeta(\mathcal{F}\Phi^t v,t)$. Two cocycles ζ,ξ are cohomologous if and only if this is true for $\mathcal{F}\zeta$ and $\mathcal{F}\xi$. In other words, $\mathcal{F}$ induces an action on cohomology classes of Hölder cocycles which we denote again by $\mathcal{F}$. We call ζ *quasi-invariant under the flip* if the cocycles ζ and $\mathcal{F}\zeta$ are cohomologous, i.e. if $[\zeta]$ is a fix point for the action of $\mathcal{F}$.

Consider now a closed negatively curved surface M. Every flip invariant Hölder cohomology class for the geodesic flow defines a unique cross ratio ([H5]), and in the two-dimensional case the defining properties of a cross ratio ([H5]) show that we can obtain from a cross ratio a finitely additive signed measure on $\mathcal{G}\tilde{M}$ in a in a natural way. In other words, there is a natural linear map which associates to a Hölder cohomology class $[\zeta]$ a signed measure ν_ζ. Now the *length cocycle* ℓ of the negatively curved manifold (M,g) is defined by $\ell(v,t)=t$ for all $v\in T^1M$ and all $t\in\mathbf{R}$. For the length cocycle ℓ of g on the surface M, ν_ℓ is just $1/2$ times the Lebesgue Liouville current of g.

On the other hand, if the Hölder cohomology class is *positive*, i.e. if it can be represented by a cocycle $\bar{\zeta}$ which satisfies $\bar{\zeta}(v,t)>0$ for all $v\in T^1M$ and all $t>0$, then there is another more classical way to define a projective class of currents on $\mathcal{G}\tilde{M}$. Namely, a suitable positive multiple $[a\zeta]$ of $[\zeta]$ (here $a>0$ is a constant depending on $[\zeta]$) defines a *Gibbs equilibrium state* which is a $\pi_1(M)$-invariant measure on $\mathcal{G}\tilde{M}$, determined uniquely up to a constant. This measure is a geodesic current if and only if $[\zeta]=[\mathcal{F}\zeta]$, and if this is satisfied we call it a *Gibbs-current of* ζ and denote its projective class in the space of projectivized currents by $[\mu_\zeta]$. The assignment $[\zeta]\to[\mu_\zeta]$ then is a map from a subset of the projectivization of the space of flip invariant cohomology classes into the projectivization of the space of geodesic currents.

If ℓ is the length cocycle of the Riemannian metric g on M, then its Gibbs current is the *Bowen-Margulis current* of g which corresponds to the measure of maximal entropy for the geodesic flow.

Katok showed in [K] that the Lebesgue Liouville current and the Bowen Margulis current of a negatively curved metric on a surface are equivalent if and only if the metric has constant curvature. With our above notation this result can be stated as follows: If ℓ is the length cocycle of a metric g of negative curvature on the surface M and if ν_ℓ is equal to μ_ℓ up to a constant, then g has constant curvature.

In Section 2 we obtain a generalization of Katok's result to arbitrary positive Hölder cohomology classes. For its formulation denote by $[\nu]$ the class of a current in the projectivization of the space of geodesic currents. We show:

Theorem C: *Let M be a closed surface of genus $g\geq 2$. If $[\mu_\ell]=[\nu_\ell]$ for a positive flip invariant Hölder cohomology class ℓ, then ℓ is the class of the length cocycle of a metric on M of constant curvature.*

Before we proceed we fix a few more notations. All our manifolds will be closed and equipped with a fixed negatively curved metric. We denote by T^1M (or $T^1\tilde{M}$) the unit tangent bundle of M (or $\tilde{M}$). There is a natural $\pi_1(M)$-equivariant projection $\pi: T^1\tilde{M}\to\partial\tilde{M}$. The pre-image of a point $\xi\in\partial\tilde{M}$ under π equals the stable manifold in $T^1\tilde{M}$ of all

directions pointing towards ξ. We let $W^i(v)$ be the leaf of the foliation W^i containing v ($i = \mathrm{ss}, \mathrm{su}$) and write P for the canonical projections $T^1M \to M$ and $T^1\tilde{M} \to \tilde{M}$.

We will need the following facts which were pointed out by Ledrappier ([L]): The fundamental group $\pi_1(M)$ of M acts naturally on the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ as a group of homeomorphisms. There is a natural bijection between the space of Hölder cocycles for the geodesic flow on T^1M and the space of Hölder cocycles for the action of $\pi_1(M)$ on $\partial\tilde{M}$. Such a Hölder cocycle for the action of $\pi_1(M)$ is a Hölder continuous functions $\zeta_0 \colon \pi_1(M) \times \partial\tilde{M} \to \mathbf{R}$ which satisfies $\zeta_0(\varphi_1\varphi_2, \xi) = \zeta_0(\varphi_1, \varphi_2\xi) + \zeta_0(\varphi_2, \xi)$ for $\varphi_1, \varphi_2 \in \pi_1(M)$ and $\xi \in \partial\tilde{M}$. Two such functions ζ_0, η_0 are *cohomologous* if there is a Hölder continuous function β_0 on $\partial\tilde{M}$ such that $\zeta_0(\varphi, \xi) - \eta_0(\varphi, \xi) = \beta_0(\varphi\xi) - \beta_0(\xi)$ for all $\varphi \in \pi_1(M)$ and all $\xi \in \partial\tilde{M}$. If $\xi(\varphi)$ is the attracting fixed point for the action of $\varphi \in \pi_1(M)$ on $\partial\tilde{M}$, then $\zeta_0(\varphi, \xi(\varphi))$ is called the *period* of ζ_0 at φ. Two Hölder cocycles are cohomologous if and only if they have the same periods.

The natural map from Hölder cocycles for the geodesic flow Φ^t to Hölder cocycles for the action of $\pi_1(M)$ on $\partial\tilde{M}$ preserves the equivalence relation which defines cohomology. Moreover a Hölder class $[\zeta]$ for Φ^t is invariant under the flip if and only if the corresponding class $[\zeta_0]$ for the action of $\pi_1(M)$ satisfies $\zeta_0(\varphi, \xi(\varphi)) = \zeta_0(\varphi^{-1}, \xi(\varphi^{-1}))$ for all $\varphi \in \pi_1(M)$ (all this is discussed in [L]). In particular, the space of Hölder classes for the geodesic flow of a negatively curved metric g on M as well as the notion of flip-invariance and positivity do not depend on the choice of the metric.

Let $\mathcal{M}$ be the space of Φ^t-invariant Borel probability measures on T^1M. For every Hölder cocycle ζ for $\Phi^t$and every $\mu \in \mathcal{M}$ the integral $\int \zeta d\mu$ can be defined. This integral only depends on the cohomology class $[\zeta]$ of ζ. More precisely, let $\alpha \colon T^1M \to \mathbf{R}$ be a Hölder cocycle and write $f(v) = \alpha(v, 1)$ for all $v \in T^1M$. Then f is a Hölder continuous function on T^1M and defines a Hölder cocycle α_f by the formula $\alpha_f(v, t) = \int_0^t f(\Phi^s v)\, ds$. If $v \in T^1M$ is a periodic point for Φ^t of period $\tau > 0$ then

$$\alpha_f(v, \tau) = \int_0^\tau f(\Phi^s v)\, ds = \int_0^\tau \alpha(\Phi^s v, 1)\, ds$$

$$= \int_0^\tau (\alpha(v, s+1) - \alpha(v, s))\, ds = \int_0^1 (\alpha(v, \tau + s) - \alpha(v, s))\, ds = \alpha(v, \tau)$$

by the cocycle equality for α. But this just means that α and α_f are equivalent (compare [L]). Thus every Hölder cohomology class can be represented by a Hölder continuous function f, and two functions f and g define the same Hölder cohomology class if and only if for every element η from the space $\mathcal{M}$ of Φ^t- invariant Borel probability measures on T^1M we have $\int f d\eta = \int g d\eta$ (this is the Livshicv theorem for Hölder continuous functions).

Denote by $[f]$ the cohomology class of the cocycle α_f defined by f. Recall that the *flip* $\mathcal{F} \colon T^1M \to T^1M$ defined by $\mathcal{F}(v) = -v$ is a diffeomorphism of T^1M which satisfies $\mathcal{F} \circ \Phi^t = \Phi^{-t} \circ \mathcal{F}$. Call $[f]$ *flip-invariant* if $[f]$ can be represented by a function g on T^1M which is invariant under $\mathcal{F}$. Again this notion is independent of the choice of a negatively curved metric on M and coincides with the notion used above.

Thus for a Hölder class ζ and a measure $\mu \in \mathcal{M}$ we can define $\int \zeta d\mu = \int \zeta(v, 1) d\mu(v)$. Then the *pressure* $pr([\zeta])$ of $[\zeta]$ is defined to be the supremum of the values $h_\mu - \int \zeta d\mu$

where μ ranges through $\mathcal{M}$ and h_μ is the *entropy* of μ. If $[\zeta]$ is *positive*, i.e. if $[\zeta]$ can be represented by a cocycle $\bar\zeta$ which satisfies $\bar\zeta(v,t) > 0$ for all $v \in T^1M$ and all $t > 0$, then there is a unique number $a > 0$ such that $[a\zeta]$ is *normalized* i.e. that $pr([a\zeta]) = 0$. The Gibbs current $[\mu_\zeta]$ of $[\zeta]$ then is the projection of the Gibbs equilibrium state of $a\zeta$ to $\mathcal{G}\tilde M$.

As was shown in [B2], for a surface M there is a natural bilinear form, the so called *intersection form*, on the compact convex space of geodesic currents for M. This bilinear form is continuous with respect to the weak*- topology on the space of geodesic currents. The intersection of the Lebesgue Liouville current of a negatively curved metric g with any current β is just twice the integral of the length cocycle of g with respect to β.

One consequence of Theorem B is the following: For every closed manifold M, the map which assigns to the class of the length cocycle of a strictly 1/4-pinched negatively curved metric on M its Lebesgue Liouville current is injective. Thus we can define the intersection between a current ν and the Lebesgue Liouville current λ_ℓ of the length cocycle ℓ of such a metric as $\int \ell d\nu$.

We conjecture that the class of the length cocycle of every negatively curved metric is determined by its Lebesgue-Liouville current, and that it is possible to extend the above function to a continuous non-symmetric bilinear form defined on the vector space spanned by all Gibbs currents.

2. Gibbs currents for surfaces

To begin with, let M be an arbitrary closed Riemannian manifold of negative sectional curvature. Recall that the space of cohomology classes of Hölder cocycles for the geodesic flow on T^1M is independent of the choice of a metric of negative curvature on M (compare [L] and the introduction).

We will only consider flip invariant Hölder cohomology classes. The cohomology class $[f]$ defined by a function f on T^1M is called *positive* if it can be represented by a positive Hölder function. This notion coincides with the one given in the introduction. Denote by $\mathcal{H}$ the space of positive flip invariant Hölder classes. This space carries a natural (non-complete) topology as follows:

Recall that a *geodesic current* for M is a $\Gamma = \pi_1(M)$-invariant locally finite Borel measure on $\partial\tilde M \times \partial\tilde M - \Delta$ where Δ is the diagonal in $\partial\tilde M \times \partial\tilde M$, which in addition is invariant under the natural involution of $\partial\tilde M \times \partial\tilde M$ exchanging the two factors. Geodesic currents correspond naturally to finite Φ^t-invariant Borel measures on T^1M which are invariant under the flip. We equip the space $\mathcal{C}$ of geodesic currents with the weak*-topology. With this topology, $\mathcal{C}$ is a locally compact space.

For every $\alpha \in \mathcal{H}$ and every current $\eta \in \mathcal{C}$ the integral $\int \alpha d\eta$ is well defined. We equip $\mathcal{H}$ with the coarsest topology such that the function $(\alpha,\eta) \in \mathcal{H}\times\mathcal{C} \to \int \alpha d\eta$ is continuous. This then corresponds to the topology of uniform convergence for continuous functions f on T^1M if we represent a current by a Φ^t-invariant finite Borel measure on T^1M and a Hölder class by a Hölder continuous function on T^1M.

Consider again the geodesic flow Φ^t on the unit tangent bundle T^1M of M. Let $\mathcal{M}$ be the compact convex space of Φ^t-invariant Borel-probability measures on T^1M equipped

with the weak*-topology. Recall that the *pressure* $pr(f)$ of a continuous function f on T^1M is defined by $pr(f) = \sup\{h_\nu - \int f d\nu \mid \nu \in \mathcal{M}\}$ where h_ν is the entropy of ν. Call a Hölder class *normalized* if it can be represented by a function f with $pr(f) = 0$. Every normalized Hölder class is positive. Vice versa, if α is a positive Hölder class, then there is a unique constant $h(\alpha) > 0$ such that $h(\alpha)\alpha$ is normalized. We call $h(\alpha)$ the *topological entropy* of α.

Identify the space of flip invariant normalized Hölder classes with the projectivization $\mathcal{PH}$ of $\mathcal{H}$.

Every element from $\mathcal{PH}$ determines its Gibbs current whose class in the space $\mathcal{PC} = \mathcal{C}/\mathbf{R}_+$ of *projective currents* where $\mathbf{R}_+$ acts on $\mathcal{C}$ via multiplication with a positive constant does not depend on any choices made. Thus we obtain a natural map of the space $\mathcal{PH}$ of projective flip invariant positive Hölder classes into the space $\mathcal{PC}$ of projective currents. We call the image of $\mathcal{PH}$ under this map the *space of projective Gibbs currents*. Also we call a current a *Gibbs current* if it corresponds to the Gibbs equilibrium state of a normalized flip invariant Hölder class on T^1M. Denote by $[\mu_\alpha]$ the class in $\mathcal{PC}$ defined by the Gibbs state of $h(\alpha)\alpha$ where $\alpha \in \mathcal{H}$ and as before, $h(\alpha)$ is the topological entropy of α.

Clearly $h(a\alpha) = a^{-1}h(\alpha)$ for all $a > 0$, moreover if α is the length cocycle of a negatively curved Riemannian metric g on M then $h(\alpha)$ is just the topological entropy of the geodesic flow of g. First of all we have:

Lemma 2.1: *$h(\alpha + \beta) \leq h(\alpha)h(\beta)/(h(\alpha) + h(\beta))$ with equality if and only if α is a constant multiple of β.*

Proof: Fix a negatively curved metric g on M with geodesic flow Φ^t. Let α, β be normalized Hölder classes. Let μ be the unique Φ^t-invariant Borel probability measure on the unit tangent bundle for (M, g) which is a Gibbs equilibrium state for the normalization of $\alpha + \beta$ and denote by h_μ the entropy of μ. Then we have $\int \alpha d\mu \geq h_\mu/h(\alpha), \int \beta d\mu \geq h_\mu/h(\beta)$ with equality if and only if $\alpha = \beta$. Then $\int[\alpha + \beta]d\mu \geq h_\mu(1/h(\alpha) + 1/h(\beta))$ and hence the lemma follows from the fact that $h_\mu = h(\alpha + \beta)\int(\alpha + \beta)d\mu$. **q.e.d.**

We equip $\mathcal{PC}$ with the topology induced from the weak*-topology on $\mathcal{C}$.

Lemma 2.2: *i) The topological entropy $h: \mathcal{H} \to (0, \infty)$ is continuous. ii) The map $[\alpha] \in \mathcal{PH} \to [\mu_\alpha] \in \mathcal{PC}$ is continuous.*

Proof: Let $\{\alpha_i\} \subset \mathcal{H}$ be a sequence converging to some $\alpha \in \mathcal{H}$. For $i \geq 1$ let $\eta_i \in \mathcal{M}$ be the unique Gibbs equilibrium state for $h(\alpha_i)\alpha_i$. Since $\mathcal{M}$ is compact we may assume by passing to a subsequence that the measures η_i converge weakly to some $\eta \in \mathcal{M}$. If h_ν denotes again the entropy of $\nu \in \mathcal{M}$ then $\nu \to h_\nu$ is upper semi-continuous and hence $h_\eta \geq \lim_{i\to\infty} \sup h_{\eta_i}$.

By the definition of the topology on $\mathcal{H}$ we have

$$\int \alpha_i d\eta_i \to \int \alpha d\eta > 0 (i \to \infty),$$

in particular $\{h(\alpha_i)\}$ is bounded from above and below by a positive constant. By passing to a subsequence we may assume that $h(\alpha_i) \to \bar{h} > 0 (i \to \infty)$. Then $h_\eta - \bar{h}\int \alpha d\eta \geq \limsup_{i\to\infty} h_{\eta_i} - h(\alpha_i)\int \alpha_i d\eta_i = 0$ and consequently $\bar{h} \leq h(\alpha)$. On the other hand, if $\bar{h} < h(\alpha)$ then there is $\nu \in \mathcal{M}$ with $h_\nu - \bar{h}\int \alpha d\nu > 0$. Then also $h_\nu - h(\alpha_i)\int \alpha_i d\nu > 0$ for i sufficiently large which is impossible. In other words, the function h is indeed continuous. Moreover, if $\eta \in \mathcal{M}$ is as above, then $h_\eta - \bar{h}\int \alpha d\eta = 0$ and hence the current determined by η is contained in the class of $[\mu_\alpha]$. From this the continuity of the map $[\alpha] \to [\mu_\alpha]$ is immediate. **q.e.d.**

Recall from [H5] that a *cross-ratio* for $\Gamma = \pi_1(M)$ is a Hölder continuous positive function Cr on the space of quadruples of pairwise distinct points in $\partial\tilde{M}$ with the following properties:

1) Cr is invariant under the action of Γ on $(\partial\tilde{M})^4$.
2) $Cr(a,a',b,b') = Cr(a',a,b,b')^{-1}$
3) $Cr(a,a',b,b') = Cr(b,b',a,a')$
4) $Cr(a,a',b,b')Cr(a',a'',b,b') = Cr(a,a'',b,b')$
5) $Cr(a,a',b,b')Cr(a',b,a,b')Cr(b,a,a',b') = 1$.

Property 5) above is a consequence of properties 1) - 4) and the fact that Cr admits a Hölder continuous extension to the space of quadruples (a,a',b,b') of points in $\partial\tilde{M}$ which satisfy $\{a,a'\} \cap \{b,b'\} = \emptyset$. This extension equals 1 for every quadruple (a,a',b,b') for which either $a = a'$ or $b = b'$ (this was communicated to me by F. Ledrappier).

We showed in [H5] that there is natural bijection between the space of cohomology classes of flip invariant Hölder cocyles and the space of cross ratios on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. We call a cross ratio Cr *positive* if it corresponds under this identification to a positive Hölder class.

Lemma 2.3: *Let Cr be a cross ratio on $\partial\tilde{M}$. Then $Cr(a,b,c,d)^{-1} + Cr(b,c,d,a)^{-1} \to 1$ as $a \to b$, locally uniformly in (c,d) if and only if Cr is positive.*

Proof: Recall from [H5] that there is a Hölder continuous symmetric function $(\ ,\ )$ on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ (where Δ denotes the diagonal) such that $\log Cr(a,b,c,d) = (a,d) + (b,c) - (a,c) - (b,d)$ for pairwise distinct points a,b,c,d in $\partial\tilde{M}$. If Cr is positive then $(a,b) \to \infty$ as $a \to b$ in $\partial\tilde{M}$. Now

$$Cr(a,b,c,d)^{-1} + Cr(b,c,d,a)^{-1} = e^{(a,c)+(b,d)}[e^{-(a,d)-(b,c)} + e^{-(b,a)-(c,d)}]$$

and this converges to 1 as $a \to b$ if and only if $e^{-(b,a)-(c,d)} \to 0$ as $a \to b$. The formula also shows that this convergence is locally uniform in (c,d) if Cr is positive. **q.e.d.**

From now on we specialize to the case that M is an oriented surface. The ideal boundary of its universal covering is naturally homeomorphic to S^1. Fix once and for all an orientation for the circle S^1. This orientation then determines for every ordered pair (a,b) of points $a \neq b$ in S^1 a unique half-open interval $[a,b[\subset S^1$ with endpoints a and b. Call a quadruple (a_1,a_2,a_3,a_4) of pairwise distinct points in S^1 *ordered* if a_i is contained in $[a_{i-1},a_{i+1}[$ for $i = 1,\dots,4$.

Proposition 2.4: *For every Gibbs current μ there is a unique cross ratio Cr for $\Gamma = \pi_1(M)$ such that* $\log Cr(a,b,c,d) = \mu[a,b[\times[c,d[$ *for every ordered quadruple* (a,b,c,d) *in* S^1. *Moreover Cr is positive.*

Proof: Let (a,b,c,d) be an ordered quadruple of pairwise distinct points in S^1 and define $[a,b,c,d] = \mu[a,b[\times[c,d[$. Since μ is a current and hence invariant under exchange of the intervals $[a,b[$ and $[c,d[$ we have $[c,d,a,b] = \mu[c,d[\times[a,b[= [a,b,c,d]$, i.e. $Cr = e^{[\,]}$ satisfies 3) above. Moreover, if $y \in]a,b[$ then (a,y,c,d) and (y,b,c,d) are ordered and

$$(*) \qquad \mu[a,b[\times[c,d[= \mu[a,y[\times[c,d[+\mu[y,b[\times[c,d[$$

which corresponds to 4) above for Cr on ordered quadruples.

So far we have not used that μ is a Gibbs current; this now is essential to show that $[\,]$ is Hölder continuous. Fix a base point $x \in \tilde{M}$ and view $S^1 = \partial\tilde{M}$ as the unit sphere in $T_x\tilde{M}$. By formula (*) it is enough to show the following: For $(c,d) \in S^1 \times S^1 - \Delta$ and $b \in]d,c[$ there are constants $\beta > 0, \alpha > 0$ such that $\mu[a,b[\times[c,d[\leq \beta|a-b|^\alpha$ whenever a is sufficiently close to b.

For this choose a positive Hölder function f on T^1M such that μ is the Gibbs current for f. Recall from [H5] that f induces a symmetric function α_f on $S^1 \times S^1 - \Delta$ such that $\beta^{-1}\alpha_f(a,b)^{1/\chi} \leq (a \mid b) \leq \beta\alpha_f(a,b)^\chi$ where $(\ \mid\ )$ is the Gromov product on $S^1 \times S^1 - \Delta$ with respect to the base-point x which defines the Hölder structure on $S^1 = \partial\tilde{M}$ and $\chi \in (0,1), \beta > 0$ are fixed constants. Moreover for a sufficiently close to $b, \mu[a,b[\times[c,d[$ is bounded from above by a constant multiple of $\alpha_f(a,b)$. From this Hölder continuity of $[\,]$ on ordered quadruples of points in $\partial\tilde{M}$ is immediate.

Next, if (a,b,c,d) is ordered, then we define $[b,a,c,d] = -[a,b,c,d]$; this then implies also that $[a,b,c,d] = -[a,b,d,c]$. Finally if we put $[b,a,d,c] = [a,b,c,d]$ whenever (a,b,c,d) is ordered then we obtain an extension of $[\,]$ to all quadruples of pairwise distinct points in $\partial\tilde{M}$ which is independent of the choice of an orientation for $\partial\tilde{M}$. Moreover $Cr = e^{[\,]}$ clearly satisfies all defining properties of a cross ratio. **q.e.d.**

We call the cross ratio Cr defined as above by a Gibbs current μ the *intersection cross ratio* of μ and we write $[\,]_\mu = \log Cr$. To justify this notion, recall that the *intersection form* i is a continuous bilinear form on the space of geodesic currents where the space of currents is equipped with the weak*-topology (see [B2]). Recall also that every free homotopy class $[\gamma]$ in M defines a unique geodesic current which we denote again by $[\gamma]$ as follows: Represent $[\gamma]$ by a closed geodesic in M. The lifts of this geodesic to $\tilde{M}$ define a $\Gamma = \pi_1(M)$-invariant subset of the space $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ of geodesics whose intersection with every compact subset of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ is finite. Then $[\gamma]$ is the sum of all Dirac masses on all points of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ corresponding to those lifts. Recall also that a free homotopy class in M is nothing else but a conjugacy class in $\pi_1(M)$.

Then we have:

Lemma 2.5: *Let μ be a Gibbs current, let $\Psi \in \Gamma$ and denote by a,b the fixed points of the action of Ψ on S^1. Then for every $\xi \in]b,a[$, the intersection of μ with the conjugacy*

class of Ψ *equals*

$$|[a, b, \xi, \Psi\xi]_\mu|.$$

Proof: By definition, if ξ is such that $(a, b, \xi, \Psi\xi)$ is ordered, then

$$[a, b, \xi, \Psi\xi]_\mu = \mu[a, b[\times[\xi, \Psi\xi[;$$

but the right hand side of this equation just equals the intersection of μ with current defined by the conjugacy class of Ψ (see [O]). **q.e.d.**

Using Lemma 2.5, we observe next that the intersection cross ratio just describes the duality between the Lebesgue-Liouville current of a metric g of negative curvature on M and the Bowen-Margulis current of the geodesic flow for g. For this we recall again that there is a natural 1-1-correspondence between flip invariant Hölder classes and cross ratios ([H5]).

Corollary 2.6: *Let* λ_g *be the Lebesgue Liouville current of a metric* g *on* M *of negative curvature. Then the intersection cross ratio of* λ_g *is twice the cross ratio defined by the length element of* g*.*

Proof: The computations in [B1] show that the intersection of the Lebesgue Liouville current λ_g of g with a free homotopy class γ in M is just twice the length of the closed geodesic with respect to g which represents γ. In other words, the value at γ of the Hölder cohomology class corresponding to the intersection cross ratio of λ_g equals twice the value at γ of the length cocycle of the metric g. Since a Hölder class is determined by its values on free homotopy classes, the corollary follows. **q.e.d.**

Another consequence of Proposition 2.4 is the following.

Lemma 2.7: *The map* $[\alpha] \in \mathcal{PH} \to [\mu_\alpha] \in \mathcal{PC}$ *is injective.*

Proof: Let $\alpha \in \mathcal{H}$ be normalized and let $\mu \in [\mu_\alpha]$. Let $\Psi \in \Gamma = \pi_1(M)$ and let a be the attracting fix point for the action of Ψ on $\partial\tilde{M}$, b be the repelling fix point. For fixed $\xi \in]b, a[$ we have $\alpha(\Psi, a) = -\lim_{k\to\infty} \frac{1}{k} \log[b, \xi, \Psi^k\xi, a]_\mu$ (see [H5]) and hence α is completely determined by $[\,]_\mu$. From this the lemma is immediate. **q.e.d.**

We can also ask for the reverse of the above procedure. Namely, let Cr be a cross ratio and write $[\,] = \log Cr$. For an ordered quadruple (a, b, c, d) of pairwise distinct points in S^1 define $\eta[a, b[\times[c, d[= [a, b, c, d]$. By our assumption, this defines a function on those subsets of $S^1 \times S^1 - \Delta$ which are products of non-empty right-half open intervals in S^1. Since $[a, y, c, d] + [y, b, c, d] = [a, b, c, d]$ for all $y \in]a, b[$, this function has a natural extension to a finitely additive function on the family of finite unions products of of right-half open intervals in S^1. If this function is in addition positive, then it defines a locally finite Borel measure on $S^1 \times S^1 - \Delta$ which we denote again by η. Since $[\,]$ is invariant under the action

of Γ , the measure η is in fact a geodesic current. We call η the *intersection current* of Cr. In general however the finitely additive function on finite unions of products of right-half open intervals in S^1 is not σ-additive, which means that it is not a *signed geodesic current* (the formal difference of two geodesic currents). However we always call η the intersection current of Cr.

The results of [H5] show that every Hölder class α on T^1M defines a unique intersection current ν_α. The assignment $\alpha \to \nu_\alpha$ is linear, i.e. if α, β are Hölder classes and if $a, b \in \mathbf{R}$ then $\nu_{a\alpha+b\beta} = a\nu_\alpha + b\nu_\beta$.

Clearly if the intersection current ν_α is positive, i.e. if ν_α is in fact a current, then α is a positive Hölder class. The reverse however is not true. To see this let α_1, α_2 be the Hölder classes of the length cocyles of hyperbolic metrics $g_1 \neq g_2$ on M. Since the metrics g_1, g_2 are bilipschitz equivalent there is a number $\epsilon > 0$ such that $\alpha_1 - \epsilon\alpha_2$ is a positive Hölder class.

By Corollary 2.6 the intersection current λ_i of α_i is $\frac{1}{2}$ times the Lebesgue Liouville current of $g_i (i = 1, 2)$. Since the currents λ_1 and λ_2 are singular, the intersection current $\lambda_1 - \epsilon\lambda_2$ of the positive Hölder class $\alpha_1 - \epsilon\alpha_2$ is not positive.

Even if the positive Hölder class α is such that its intersection current ν_α is positive and ergodic under the action of $\Gamma = \pi_1(M)$ we do not know whether ν_α is in fact a Gibbs current.

For every geodesic current η and every Hölder class α the integral $\int \alpha d\eta$ is well defined. More precisely, recall from [B2] that for two currents η, η' the intersection $i(\eta, \eta')$ is defined. Then we have:

Lemma 2.8: *For every positive Hölder class α for which ν_α is positive and every current β we have $\int \alpha d\beta = i(\beta, \nu_\alpha)$.*

Proof: Let γ be a current defined by a conjugacy class in $\pi_1(M)$. Let $\Psi \in \pi_1(M)$ be contained in this conjugacy class and let a, b be the fixed points of the action of Ψ on S^1. Then $i(\gamma, \beta) = \beta]a, b[\times[\xi, \Psi\xi[$ for every $\xi \in S^1 - \{a, b\}$ and every current β and hence by the definition of the map $\alpha \to \nu_\alpha$ (see [H5]) we have $i(\gamma, \nu_\alpha) = \int \alpha d\gamma$. Since finite sums of weighted Dirac currents of conjugacy classes in $\pi_1(M)$ are dense in the space of all currents equipped with the weak*-topology and since the intersection form is continuous with respect to the weak*-topology ([B2]) the lemma follows. **q.e.d.**

Every Riemannian metric g on M of negative curvature defines its Lebesgue-Liouville current λ_g.

Lemma 2.9: *The set of Lebesgue Liouville currents of negatively curved metrics is dense in the set of Gibbs currents.*

Proof: Since every Gibbs current can be approximated by a sequence of Dirac currents for conjugacy classes in $\pi_1(M)$ it suffices to show that the closure of the set of all Lebesgue Liouville currents of metrics of negative curvature in the space $\mathcal{C}$ of geodesic currents equipped with the weak*-topology contains all currents defined by conjugacy classes in $\pi_1(M)$.

For this recall first of all that a free homotopy class in M which can be represented by a simple closed curve defines the projective class of a measured geodesic lamination. On the other hand, projective measured laminations form the boundary in $\mathcal{PC}$ of the projectivizations of the Lebesgue Liouville currents of hyperbolic metrics on M ([B1]). Thus we may restrict our attention to free homotopy classes which can not be represented by simple closed curves.

Fix a hyperbolic metric g on M and let γ be a closed geodesic in (M, g) with self intersections. By eventually changing the metric g we may assume that γ has only double points.

For simplicity we consider only the case that γ has exactly one double point. The construction given below is also valid in the general case. Let $\gamma: [0, T] \to M$ be a parametrization by arc length such that $\gamma(0) = \gamma(\tau) = \gamma(T)$ for some $\tau \in (0, T)$ (in other words, $\gamma(0)$ is the double point).

For $\epsilon > 0$ let $Q(\epsilon)$ be the ϵ-neighborhood of γ in M. For sufficiently small ϵ, $Q(\epsilon) - \gamma[0, \tau]$ has exactly two components, one of which, say Q_1, is homeomorphic to an annulus. Choose a normal vector field $t \to N(t)$ along γ which points on $(0, \tau)$ inside Q_1. Let $m > 0$ be a large number; then there are $\epsilon_1(m), \epsilon_2(m) \in (0, \epsilon)$ such that $\exp\, \epsilon_1(m)N(1/m) = \exp\, \epsilon_1(m)N(\tau - 1/m) \in Q_1$ (where exp is the exponential map of (M, g)) and $\exp\,(-\epsilon_2(m)N(1/m)) = \exp\, \epsilon_2(m)N(\tau + 1/m))$. Observe that the absolute values of $\epsilon_1(m)$ and $\epsilon_2(m)$ only depend on m and the angle between $\gamma'(0)$ and $-\gamma'(\tau)$ at $\gamma(0)$.

Thus for each sufficiently large $m \geq 1$ we obtain a closed neighborhood $A(m)$ of γ in M with the following properties:

i) $A(m) \supset A(m + 1)$ and $\cap_{m \geq 1} A(m) = \gamma$

ii) $A(m) = \cup_{i=0}^{4} A_i(m)$ where the sets $A_i(m)$ are closed with piecewise smooth boundary and pairwise disjoint interior.

iii) $A_0(m)$ is a geodesic quadrangle in M containing $\gamma(0)$ in its interior, with vertices

$$\exp\, \epsilon_1(m)N(1/m) = x_1(m),\ \exp\, -\epsilon_2(m)N(1/m) = x_2(m),$$
$$\exp\,(-\epsilon_1(m)N(T - 1/m)) = x_3(m) \text{ and } \exp\, \epsilon_2(m)N(T - 1/m) = x_4(m).$$

iv) The boundary of $A_i(m)(i = 1, \ldots, 4)$ contains two smooth geodesic segments of length $\epsilon_1(m)$ or $\epsilon_2(m)$ which meet at $x_i(m)$ and are subarcs of the boundary of $A_0(m)$. It also contains a subarc of γ. With respect to normal exponential coordinates based at γ the metric on $A_i(m)$ can be written in the form $(\cosh s)^2 dt^2 + ds^2 (s \in [0, \epsilon_1(m)]$ or $s \in [0, \epsilon_2(m)])$.

Change now the metric in the interior of $A(m)$ as follows: Fix $m \geq 1$ sufficiently large and for $i = 1, 2$ choose a diffeomorphism $\Psi_{i,m}: [0, m] \to [0, \epsilon_i(m)]$ with the following properties:

1) $\Psi'_{i,m}(s) = 1$ for s near 0 and s near m.
2) $\frac{-\Psi''_{i,m}(s)}{\Psi'_{i,m}(s)} < \frac{\cosh(\Psi_{i,m}(s))}{\sinh(\Psi_{i,m}(s))}$ for $s \in [0, m]$

Property 2) can be fulfilled since $\frac{\cosh(s)}{\sinh(s)} \cdot s \to 1$ as $s \searrow 0$. Define now a new metric g_m on M as follows:

a) g_m coincides with the original metric g outside $A(m)$.
b) In normal exponential coordinates based at γ the restriction of g_m to $A_j(m)$ can be written in the form $dg_m = (\cosh s)^2 dt^2 + [(\Psi_{i,m}^{-1})'(s)]^2 ds^2 (i = 1$ or $i = 2$ and $j = 1, \dots, 4)$.

This defines g_m on $M - A_0(m)$. Observe that g_m has negative curvature (this is guaranteed by property 2) for $\Psi_{i,m}$) and has a natural extension to the boundary of A_0 which coincides with g near the vertices $x_j(m)(j = 1, \dots, 4)$. With respect to this extension, A_0 is a geodesic quadrangle of side length $2m$ and such that the sum of the internal angles of the quadrangle is strictly less than 2π. This means that we can extend g_m to a metric of negative curvature on $A_0(m)$ in such a way that as $m \to \infty$ the geodesic quadrangle $(A_0(m), g_m)$ approaches an euclidean parallelogram of side length $2m$ whose angles are determined by the angle between $\gamma'(0)$ and $-\gamma'(\tau)$. Moreover we may assume that γ remains a g_m-geodesic for all m.

Let λ'_m be the Lebesgue Liouville current of g_m and write $\lambda_m = \frac{1}{4m}\lambda'_m$. Choose a free homotopy class in M which is prime, different from the class of γ and which is represented by the g-geodesic η. By abuse of notation we use the same symbol for η and its free homotopy class.

After a slight perturbation of η we may assume that η does not pass through $\gamma(0)$. Then η intersects γ in finitely many points $\eta(s_1), \dots, \eta(s_q)$ where $q = i(\gamma, \eta)$. For sufficiently large $m > 1$ the intersection of η with $A(m)$ has then precisely q connected components, and the g_m-length of each of these components is contained in the interval $[2m, 2m + c]$ where $c > 0$ is a constant which depends on η but not on m. Thus $i(\lambda_m, \eta) = \frac{1}{4m} i(\lambda'_m, \eta) \leq \frac{1}{2m} \cdot g_m -$ length of $\eta \to q(m \to \infty)$.

In other words, we have $\limsup_{m\to\infty} i(\lambda_m, \eta) \leq i(\gamma, \eta)$.

Since the intersection form i on $\mathcal{C} \times \mathcal{C}$ is continuous and since the set of finite weighted sums of Dirac currents of free homotopy classes different from γ is dense in $\mathcal{C}$ we conclude that the sequence $\{\lambda_m\}$ is bounded in $\mathcal{C}$. By passing to a subsequence we may therefore assume that $\{\lambda_m\}$ converges weakly to a current $\lambda \in \mathcal{C}$. Then $i(\lambda, \eta) \leq i(\gamma, \eta)$ for every free homotopy class $\eta \neq \gamma$ and therefore $\lambda \leq \gamma$. On the other hand the current γ is ergodic and hence $\lambda = a\gamma$ for some $a \geq 0$.

We are left with showing that $a \neq 0$. For this choose again a free homotopy class η with $i(\gamma, \eta) = q > 0$. Then every curve in M representing η intersects γ in at least q points, where an intersection at the point $\gamma(0)$ has to be counted twice. But this means that the g_m-length of every such curve is at least $2m$ and hence $\liminf_{m\to\infty} i(\lambda_m, \eta) = i(\lambda, \eta) > 0$. This shows $a \neq 0$ and finishes the proof of the lemma. **q.e.d.**

To summarize the considerations in the beginning of this chapter we have the following situation: To every element $[\alpha]$ in the space $\mathcal{PH}$ of projective flip invariant positive Hölder classes we can associate the projective class $[\mu_\alpha]$ of the Gibbs equilibrium state defined by α and also the projective class $[\nu_\alpha]$ of the intersection current of α. This defines two

injective continuous maps $[\alpha] \to [\nu_\alpha]$ and $[\alpha] \to [\mu_\alpha]$ of $\mathcal{PH}$ into the space of projective classes of finitely additive signed measures on $\mathcal{G}\tilde{M}$. The image of $\mathcal{PH}$ under the second map is just the space of projective Gibbs currents.

With this terminology, the result of Katok in [K] can be formulated as follows: If α is the class of the length cocycle of a metric on M of negative curvature, then $[\mu_\alpha] = [\nu_\alpha]$ if and only if this metric is of constant negative curvature.

Recall the definition of the *topological entropy* $h(\alpha)$ of a positive Hölder class α.

The proof of Katok uses another result proved in the same paper which can be slightly generalized as follows (for a generalization to higher dimensions see [B-C-G]):

Theorem 2.10: *Let M be a compact surface of Euler characteristic $\chi(M) < 0$ and let α be a positive Hölder class such that ν_α is a Gibbs current with $i(\nu_\alpha, \nu_\alpha) = \pi^2|\chi(M)|$. Then $h(\alpha) \geq \frac{1}{4}$ with equality if and only if α is the length cocycle of a hyperbolic metric on M.*

Proof: Let $g, \tilde{g}$ be conformally equivalent Riemannian metrics on M of negative curvature, i.e. there is a function $a: M \to (0, \infty)$ such that $ag = \tilde{g}$. Assume that the g-volume of M and the $\tilde{g}$-volume equals $(2\pi)^{-1}$. If $\lambda_g, \lambda_{\tilde{g}}$ denotes the Lebesgue Liouville current of $g, \tilde{g}$ then this just means that $i(\lambda_g, \lambda_g) = 2 = i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}})$ (see [B1]).

Let V be the unit tangent bundle of (M, g) and let λ be the Lebesgue Liouville measure of g on V. For a g-unit vector $v \in V$ denote by $\rho(v)$ the $\tilde{g}$-norm of v. If $P: V \to M$ denotes the canonical projection then $\rho(v) = (a(Pv))^{1/2}$ and consequently

$$\int_V \rho d\lambda = \int_V (a^{1/2} \circ P) d\lambda \leq (\int (a \circ P) d\lambda)^{1/2} = 1$$

with equality if and only if $a \equiv 1$.

Let now $\alpha(g), \alpha(\tilde{g})$ be the length cocycles of the metrics $g, \tilde{g}$. Since clearly $\int \alpha(\tilde{g})\, d\lambda_g \leq \int_V \rho\, d\lambda$, Lemma 2.8 shows that

$$(*) \qquad i(\lambda_g, \nu_{\alpha(\tilde{g})}) \leq 1,\ i(\lambda_{\tilde{g}}, \nu_{\alpha(g)}) \leq 1$$

with equality if and only if $g = \tilde{g}$.

Assume now in addition that $\tilde{g}$ is a metric of constant curvature and let $h(g), h(\tilde{g})$ be the topological entropy of the geodesic flow for $g, \tilde{g}$. Then the cocycles $\alpha(g)h(g)$ and $\alpha(\tilde{g})h(\tilde{g})$ are normalized. Now $\lambda_{\tilde{g}} = 2\nu_{\alpha(\tilde{g})}$, moreover the Lebesgue Liouville measure of $\tilde{g}$ equals the Gibbs equilibrium state for $\alpha(\tilde{g})h(\tilde{g})$ and therefore

$$(**) \qquad \begin{aligned} &\int \alpha(g)h(g)\, d\lambda_{\tilde{g}} = h(g)i(\lambda_{\tilde{g}}, \nu_{\alpha(g)}) \geq \\ &\int h(\tilde{g})\alpha(\tilde{g})\, d\lambda_{\tilde{g}} = h(\tilde{g})i(\lambda_{\tilde{g}}, \nu_{\alpha(\tilde{g})}) = \\ &\qquad h(\tilde{g})\frac{1}{2}i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = h(\tilde{g}). \end{aligned}$$

Together with (*) this shows that $h(g) \geq h(\tilde{g})$ with equality if and only if $g = \tilde{g}$.

Notice that $h(\tilde{g})$ does not depend on the metric $\tilde{g}$ of constant curvature with $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ (compare [B1]). We write $h > 0$ for this common number. Let now α be a positive Hölder class such that $\nu = \nu_\alpha$ is a Gibbs current and that $i(\nu_\alpha, \nu_\alpha) = \frac{1}{2}$. By Lemma 2.9, Lebesgue Liouville currents of negatively curved metrics are dense in the set of Gibbs currents. This means that there is a sequence $\{g_j\}_j$ of negatively curved metrics on M with Liouville currents λ_{g_j} and such that $\lambda_{g_j} \to 2\nu$. Since the intersection form is continuous we may assume that g_j is normalized in such a way that $i(\lambda_{g_j}, \lambda_{g_j}) = 2$.

Let α_j be the Hölder class determined by the length cocycle of the metric g_j. Since the intersection form i is continuous on the space of geodesic currents we obtain from Lemma 2.8 and the fact that $\nu_{\alpha_j} \to \nu_\alpha$ weakly in $\mathcal{C}$ that $\alpha_j \to \alpha$ in $\mathcal{H}$. Lemma 2.2 then shows that $h(\alpha_j) \to h(\alpha)$ and therefore from the above consideration we conclude that $h(\alpha) \geq h(\tilde{g}) = h$ where $\tilde{g}$ is a metric of constant curvature with $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$.

Assume now that $h(\alpha) = h$. Let $\tilde{g}_j$ be a metric of constant curvature on M which is conformally equivalent to g_j and such that $i(\lambda_{\tilde{g}_j}, \lambda_{\tilde{g}_j}) = 2$. By our assumptions we have $h(g_j) - h(\tilde{g}_j) \to 0 (j \to \infty)$. By (*) and (**) above, this means that $i(\lambda_{g_j}, \nu_{\alpha(\tilde{g}_j)}) \to 1 (j \to \infty)$.

Recall that the space of projective currents $\mathcal{PC}$ is compact; hence by passing to a subsequence we may assume that the projective classes $[\lambda_{\tilde{g}_j}]$ of $\lambda_{\tilde{g}_j}$ converge weakly to a projective current $[\beta]$. Then $\{\tilde{g}_j\}$ is an unbounded sequence in Teichmüller space if and only if β is a geodesic lamination, i.e. it satisfies $i(\beta, \beta) = 0$ ([B1]).

On the other hand, $\lambda_{g_j} \to \nu_\alpha$ and ν_α is a Gibbs current, i.e. its intersection with every current is positive. If $b_j > 0$ is such that $b_j \lambda_{\tilde{g}_j} \to \beta \neq 0$ then $i(\lambda_{g_j}, b_j \lambda_{\tilde{g}_j}) = b_j i(\lambda_{g_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \beta) > 0\, (j \to \infty)$, again by continuity of the intersection form. But $i(\lambda_{g_j}, \lambda_{\tilde{g}_j}) \to 2 (j \to \infty)$ and consequently $\{b_j\}$ is bounded from below by a positive number. This implies in turn that $\{\tilde{g}_j\}$ is bounded in Teichmüller space and hence that $[\lambda_{\tilde{g}_j}] \to [\lambda_{\tilde{g}}]$ for some metric $\tilde{g}$ of constant curvature.

If we normalize $\lambda_{\tilde{g}}$ in such a way that $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ then $\lambda_{\tilde{g}_j} \to \lambda_{\tilde{g}}$ and also $\lambda_{g_j} \to \lambda_{\tilde{g}}$ and consequently $\nu_\alpha = \frac{1}{2} \lambda_{\tilde{g}}$. This finishes the proof of the theorem. **q.e.d.**

If α is a positive Hölder class such that ν_α is a Gibbs current, then we can also define the *metric entropy* $h_m(\alpha)$ of α as follows: Let β be the unique normalized positive Hölder class such that $[\mu_\beta] = [\nu_\alpha]$ and define $h_m(\alpha) = \int \beta d\nu_\alpha / i(\nu_\alpha, \nu_\alpha) = i(\nu_\alpha, \nu_\beta)/i(\nu_\alpha, \nu_\alpha)$.

If α is the length cocycle of a metric g on M of negative curvature, then $h_m(\alpha)$ equals the metric entropy of the geodesic flow of g. Moreover $h_m(a\alpha) = a^{-1} h_m(\alpha)$ and if $[\nu_\alpha] = [\mu_\alpha]$ then $h(\alpha) = h_m(\alpha)$.

In analogy to Theorem 2.10 we can also estimate $h_m(\alpha)$:

Theorem 2.11: *Let α be a positive Hölder class for the surface M such that $i(\nu_\alpha, \nu_\alpha) = \pi^2 |\chi(M)|$ and that ν_α is a Gibbs current. Then $h_m(\alpha) \leq \frac{1}{4}$ with equality if and only if α is the length cocycle of a hyperbolic metric on M.*

Proof: As in the proof of Theorem 2.10, let $g, \tilde{g}$ be conformally equivalent Riemannian metrics with Lebesgue Liouville currents $\lambda_g, \lambda_{\tilde{g}}$ and length cocycles $\alpha(g), \alpha(\tilde{g})$ and such that $i(\lambda_g, \lambda_g) = i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2, \tilde{g}$ of constant curvature. Since the cocycle $h(\alpha(\tilde{g}))\alpha(\tilde{g})$ is normalized we have $\int h(\alpha(\tilde{g}))\alpha(\tilde{g}) d\lambda_g = h(\alpha(\tilde{g})) i(\lambda_g, \nu_{\alpha(\tilde{g})}) \geq h_m(\alpha(g)) \cdot \frac{1}{2} i(\lambda_g, \lambda_g) =$

$h_m(\alpha(g))$. But $i(\lambda_g, \nu_{\alpha(\tilde{g})}) \leq 1$ with equality only if $g = \tilde{g}$ by the proof of Theorem 2.10. Thus we obtain the statement of the theorem in the case that α is the class of a length cocycle for a negatively curved Riemannian metric. The general case now follows as in the proof of Theorem 2.10.

Namely, let α be such that ν_α is a Gibbs current and $i(\nu_\alpha, \nu_\alpha) = \frac{1}{2}$. Let α_j be the Hölder class of the length cocycle of a metric g_j on M such that $i(\nu_{\alpha_j}, \nu_{\alpha_j}) = \frac{1}{2}$ and $\nu_{\alpha_j} \to \nu_\alpha$. Choose a metric $\tilde{g}_j$ on M which is conformally equivalent to g_j and satisfies $i(\lambda_{\tilde{g}_j}, \lambda_{\tilde{g}_j}) = 2$; then $i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \leq 1$ for all j.

Now if $\tilde{g}_j$ is unbounded in Teichmüller space, then there is a sequence $\{b_j\} \subset [0, \infty)$ such that $b_j \to 0$ and that $\lambda_{\tilde{g}_j} b_j$ converges weakly to a measured lamination β. Then $i(\nu_{\alpha_j}, b_j \lambda_{\tilde{g}_j}) = b_j i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \beta) > 0$ which is impossible. Thus the sequence $\{\lambda_{\tilde{g}_j}\}$ is bounded and by passing to a subsequence we may assume that $\lambda_{\tilde{g}_j} \to \lambda_{\tilde{g}}$ $(j \to \infty)$ where $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ and $\tilde{g}$ is a metric of constant curvature. Then $i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \lambda_{\tilde{g}}) \leq 1$ and hence $h_m(\alpha) \leq h(\alpha(\tilde{g}))$ by the above argument. The discussion of equality is exactly the same as the discussion of the equality case in the proof of Theorem 2.10. **q.e.d.**

Now if again α is a positive Hölder class such that $[\nu_\alpha] = [\mu_\alpha]$, then in particular ν_α is a Gibbs current and hence Theorem 2.10 and 2.11 combined show:

Corollary 2.12: $[\mu_\alpha] = [\nu_\alpha]$ *for a positive Hölder class* α *if and only if* α *is the class of the length cocycle of a metric on* M *of constant negative curvature.*

Remark: In his paper [K], Katok showed Corollary 2.12 for length cocycles of negatively curved Riemannian metrics. The arguments given here are more formal and technically much easier, but follow the same principal idea as the argument of Katok. If M is a compact rank 1 locally symmetric space, then the analogue to Theorem 2.10 for length cocycles of negatively curved metrics is due to Besson, Courtois and Gallot ([B-C-G]). However our simple estimate using conformal equivalence of Riemannian metrics on M is not valid any more and the generalization in our Theorem 2.10 for arbitrary Hölder classes is not true in higher dimensions (compare Section 3). Moreover, the analogue of Theorem 2.11 fails even in the case that the dimension of M equals 3, as was pointed out by Flaminio ([F]). This is however not so surprising given the fact that with the notation of the proof of Theorem 2.11 we have $\int_V \rho d\lambda > \int_V d\lambda$ whenever $g \neq \tilde{g}$ and g and $\tilde{g}$ are metrics with the same volume form. However the analogue of Corollary 2.12 may well be true for manifolds which carry a locally symmetric metric.

Recall the following result of Bonahon ([B1]): If μ is a geodesic current on a surface of negative Euler characteristic such that

$$e^{-\mu[a,b]\times[c,d]} + e^{-\mu[b,c]\times[d,a]} = 1$$

for every ordered quadruple (a, b, c, d) then μ is the Lebesgue Liouville current of a hyperbolic metric. This gives a purely algebraic description of cross ratios "of maximal entropy". We can use the above considerations to give a slightly sharper version of this result.

Corollary 2.13: *Let μ be a geodesic current such that there is a number $\rho > 1$ with $\rho^{-1}e^{-\mu[a,b]\times[c,d]} \leq 1-e^{-\mu[b,c]\times[d,a]} \leq \rho e^{-\mu[a,b]\times[c,d]}$ for every ordered quadruple (a,b,c,d) in $\partial\tilde{M}$. Then μ is the Lebesgue-Liouville current of a metric of constant negative curvature.*

Proof: Let μ be a geodesic current which satisfies the assumptions in the corollary. For two points $a \neq b \in S^1$ we then can define a measure μ_{ab} on $]b,a[\subset S^1$ by $\mu_{ab}[x,y[= \mu[a,b[\times[x,y[$. We want to show that the measure class of μ_{ab} does not depend on $a \neq b$.

For this choose $c \in]a,b[$, let $x \in]b,a[$ and let $y \in]b,a[$ be a point near x. By our assumption there is a neighborhood U of x in $]b,a[$ such that

$$1 - e^{-\mu_{cb}[x,y[} = 1 - e^{-\mu[c,b[\times[x,y[} \geq \rho^{-1}e^{-\mu[b,x[\times[y,c[}$$

for all $y \in U - \{x\}$ and moreover

$$e^{-\mu[b,x[\times[y,c[} = e^{-\mu[b,x[\times[y,a[}e^{-\mu[b,x[\times[a,c[} \geq \rho^{-1}(1 - e^{-\mu[x,y[\times[a,b[})e^{-\mu[b,x[\times[a,c[}.$$

For y sufficiently close to x we have

$$\begin{aligned}1 - e^{-\mu_{cb}[x,y[} &= \mu_{cb}[x,y[+ o(\mu_{cb}[x,y[),\\ 1 - e^{-\mu_{ab}[x,y[} &= \mu_{ab}[x,y[+ o(\mu_{ab}[x,y[)\end{aligned}$$

and consequently the measures μ_{cb} and μ_{ab} are absolutely continuous at x with Radon Nikodym derivative

$$1 \geq \frac{d\mu_{cb}}{d\mu_{ab}}(x) \geq \rho^{-2}e^{-\mu[b,x[\times[a,c[}.$$

This means that the current μ, viewed as a Φ^t-invariant Borel measure on the unit tangent bundle T^1M of M, is absolutely continuous with respect to the stable foliation, with locally bounded Radon Nikodym derivative.

Let α be the Hölder class determined by μ. Since μ is a current, α is positive and is determined by its values on periodic points for the geodesic flow on T^1M. More precisely, for every $\Psi \in \pi_1(M)$ with attracting fix point a, repelling fix point b for its action on S^1 and every $\xi \in]b,a[$ and $k \geq 1$ we have $\alpha(\Psi,a) = \frac{1}{k}\mu[a,b[\times[\xi,\Psi^k\xi[$ (where we view α as a function on $\pi_1(M) \times \partial\tilde{M}$).

On the other hand, $\lim_{k\to\infty}\frac{1}{k}\log\mu[b,\xi[\times[\Psi^k\xi,a[$ is just the asymptotic logarithmic decay of the conditional measure $\mu_{b\xi}$ at the attracting fix point a under the action of Ψ. For sufficiently large $k \geq 1$ we conclude that

$$\begin{gathered}\rho^{-1}e^{-k\alpha(\Psi,a)} \leq 1 - e^{-\mu[b,\xi[\times[\Psi^k\xi,a[} = \\ \mu[b,\xi[\times[\Psi^k\xi,a[+ o(\mu[b,\xi[\times[\Psi^k\xi,a[) \leq \rho e^{-k\alpha(\Psi,a)}\end{gathered}$$

and therefore $-\alpha(\Psi,a) = \lim_{k\to\infty}\frac{1}{k}\log\mu[b,\xi[\times[\Psi^k\xi,a[$. But this just means that $[\mu] = [\nu_\alpha] = [\mu_\alpha]$ and hence the corollary follows from Corollary 2.12. **q.e.d.**

3. Contact cocycles and Lebesgue measures in higher dimensions

In Section 2 we looked at signed geodesic currents for a closed hyperbolic surface defined by the cross ratio of a Hölder class. This assignment can be considered as associating to the (de Rham) cohomology class of a Hölder continuous 1-form on the unit tangent bundle of this surface its derivative, viewed as a volume form on the space of geodesics.

In this section we discuss an analogue of this in higher dimensions: Namely the Lebesgue-Liouville current of a negatively curved metric g on a closed manifold M can always be obtained in a purely conbinatorical way from the length cocycle of g. As a corollary we obtain Theorem B from the introduction. For this we exploit the fact that the cross ratio of the length cocycle can be viewed as a coarse version of the symplectic structure on the space of geodesics.

To begin with, consider the vector space $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$, equipped with some smooth symplectic form ρ. Assume that for every fixed $x \in \mathbf{R}^n$ the sets $\{x\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{x\}$ are Lagrangian submanifolds of $(\mathbf{R}^{2n}, \rho)$. We define a function φ on $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$ as follows: For $x, y \in \mathbf{R}^n$ let $c_x\colon [0,1] \to \mathbf{R}^n$ be a smooth curve joining $c_x(0) = 0$ to $c_x(1) = x$ and choose similarly a curve $c_y\colon [0,1] \to \mathbf{R}^n$ joining 0 to y. Define a map $\Psi\colon [0,1]^2 \to \mathbf{R}^{2n}$ by $\Psi(s,t) = (c_x(s), c_y(t))$ and let $\varphi(x,y) = \int_{\Psi[0,1]^2} \rho$.

Lemma 3.1: *The function φ does not depend on the choice of the curves c_x, c_y.*

Proof: Choose a 1-form ζ such that $d\zeta = \rho$. Let $c_x, \tilde{c}_x$ and $c_y, \tilde{c}_y$ be curves joining 0 to x and y and let $\Psi, \tilde{\Psi}\colon [0,1]^2 \to \mathbf{R}^{2n}$ be the corresponding maps defined as above.

By Stoke's theorem we have $\int_{\Psi[0,1]^2} \rho = \int_{\partial\Psi[0,1]^2} \zeta$ and $\int_{\tilde{\Psi}[0,1]^2} \rho = \int_{\partial\tilde{\Psi}[0,1]^2} \zeta$. Now the oriented boundary $\partial\Psi - \partial\tilde{\Psi}$ consists of 4 loops (one of them is $c_x \circ \tilde{c}_x^{-1}$) contained entirely in a Lagrangian submanifold for ρ. Another application of Stokes's theorem then shows

$$\int_{\partial\tilde{\Psi}[0,1]^2} \zeta = \int_{\partial\Psi[0,1]^2} \zeta.$$

q.e.d.

If ρ is the standard symplectic form $\rho_0 = \sum dx_i \wedge dy_i$ on $\mathbf{R}^n \times \mathbf{R}^n$, then we write φ_0 for the function as in Lemma 3.1.

Consider now $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$ with the standard form ρ_0 and the corresponding function φ_0. For a product set $Q_1 \times Q_2 \subset \mathbf{R}^n \times \mathbf{R}^n$ define the *symplectic diameter* $\delta(Q_1 \times Q_2)$ by $\delta(Q_1 \times Q_2) = \sup\{|\varphi_0(x,y)| \mid x \in Q_1, y \in Q_2\}$ and for a compact subset K of $\mathbf{R}^n$ and $r \geq 0$ let $B(K,r) = \{y \in \mathbf{R}^n \mid |\varphi_0(x,y)| \leq r \text{ for all } x \in K\}$. Clearly we have $B(K,r) = rB(K,1)$, moreover $B(K,r)$ is star-shaped about 0 and invariant under reflection at the origin. Moreover we have:

Lemma 3.2: *Let $K \subset \mathbf{R}^n$ be a compact neighborhood of* 0 *with dense interior. Then $B(K,1)$ is a compact convex body in $\mathbf{R}^n$ which is reflection-symmetric at the origin. For*

every $x \in \mathbf{R}^n$ contained in the boundary of $B(K,1)$ a hyperplane $H \subset \mathbf{R}^n$ through x is supporting for $B(K,1)$ if and only if there is $z \in K$ with $\varphi_0(z, H-x) = 0$ and such that $|\varphi_0(z,x)| = \max\{|\varphi_0(y,x)| \mid y \in K\}$.

Proof: For $x \in \mathbf{R}^n$ write $\alpha(x) = \sup\{|\varphi_0(z,x)| \mid z \in K\}$. Since by assumption K is a compact neighborhood of 0 in $\mathbf{R}^n$ we have $\alpha(x) > 0$ for $x \neq 0$ and hence $B(K,1)$ is compact with non-empty interior. Moreover $B(K,1)$ is clearly star-shaped about 0.

Let $x \in B(K,1)$ be a point from the boundary of $B(K,1)$. Since K is compact there is $z \in K$ such that $|\varphi_0(z,x)| = \alpha(x) = 1$. Define $H(x) = \{y \in \mathbf{R}^n \mid \varphi_0(z,y) = 0\}$. Then $H(x)$ is a hyperplane in $\mathbf{R}^n$ and $B(K,1) \subset W(x) = \{tx + y \mid -1 \leq t \leq 1,\ y \in H(x)\}$.

But this just means that $H(x)$ is a supporting hyperplane for $B(K,1)$ at x. Since $W(x)$ is convex and $B(K,1) = \bigcap_{x \in \partial B(K,1)} W(x)$ we conclude that $B(K,1)$ is convex.

For a dense set of points in the boundary of $B(K,1)$ a supporting hyperplane is unique. Therefore by continuity we conclude that the set of supporting hyperplanes for $B(K,1)$ at a point $x \in \partial B(K,1)$ is in $1-1-$correspondence with pairs of points $\pm z$ such that $z \in K$ and that $|\varphi_0(z,x)| = \alpha(x)$. This shows the lemma. **q.e.d.**

Let now $K \subset \mathbf{R}^n$ be a compact set with non-empty interior and write $B_1 = B(K,1)$, $B_2 = B(B(K,1),1)$. By definition, $K \subset B_2$ and $|\varphi_0(x,y)| \leq 1$ for all $x \in B_1, y \in B_2$. Moreover B_1 and B_2 are compact convex bodies in $\mathbf{R}^n$, reflection symmetric at the origin. Now the symplectic form ρ_0 defines a linear isomorphism of $\mathbf{R} = \mathbf{R}^n \times \{0\}$ onto the dual of $\mathbf{R}^n = \{0\} \times \mathbf{R}^n$, and with respect to this identification $B_2 \subset (\mathbf{R}^n)^*$ is just the dual of the convex body $B_1 \subset \mathbf{R}^n$. We call $(B_1, B_2) \subset \mathbf{R}^n \times \mathbf{R}^n$ a *dual pair of convex bodies* in $\mathbf{R}^n$. If B_1 is an ellipsoid, i.e. if B_1 is the image of the unit ball in $\mathbf{R}^n$ (equipped with the standard inner product) under a linear isomorphism of $\mathbf{R}^n$, then we call (B_1, B_2) an *euclidean dual pair*.

Now recall that the n-th exterior power ρ_0^n of ρ_0 is a volume form on $\mathbf{R}^n$. Denote by $a(n) > 0$ the euclidean volume of the euclidean unit ball in $\mathbf{R}^n$. The following is known as the *Blaschke-Santaló inequality*, it can be found in [M-P]. (I am grateful to V. Bangert for this reference).

Proposition 3.3: *$\rho_0^n(B_1 \times B_2) \leq a(n)^2$ for every dual pair (B_1, B_2) of convex bodies in $\mathbf{R}^n$, with equality if and only if (B_1, B_2) is an euclidean dual pair.*

View $\mathbf{R}^n$ as an euclidean vector space, equipped with the inner product $<,>$. For $r > 0$ let $B(r)$ be the closed ball of radius r about 0 in $\mathbf{R}^n$; then $B(r) \times B(r) = D(r)$ is a neighborhood of 0 in $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$. For $c \geq 1$ define a *c-quasisymplectic r-ball* in $\mathbf{R}^{2n}$ to be the image of $D(r)$ under a map $\beta : D(r) \to \mathbf{R}^{2n}$ with the following properties:

1) There are continuous maps $\beta_i : B(r) \to \mathbf{R}^n (i = 1,2)$ such that $\beta = (\beta_1, \beta_2)$.
2) $|\ \varphi_0(x,y) - \varphi_0(\beta_1 x, \beta_2 y)\ | \leq (c-1)r^2$ for all $x, y \in B(r)$.

The above discussion indicates that a 1-quasisymplectic 1-ball in $\mathbf{R}^{2n}$ is an *euclidean dual pair*, i.e. there is a linear isomorphism $L : \mathbf{R}^n \to \mathbf{R}^n$ such that $\beta_1 = L$ and $\beta_2 = (L^t)^{-1}$ where L^t is the transpose of L with respect to the duality $\mathbf{R}^n \to (\mathbf{R}^n)^*$ defined by the symplectic form ρ_0.

The next lemma contains a more precise statement of the relation between quasisymplectic balls and euclidean dual pairs.

Proposition 3.4: *For every $n \geq 1$ there is a number $q(n) > 0$ with the following property: For every $\epsilon \in (0, 2^{-n})$, every $r > 0$ and every $(1+\epsilon)-$quasisymplectic r-ball D in $\mathbf{R}^{2n}$ there is an euclidean dual pair $(B_1, B_2) \subset \mathbf{R}^n \times \mathbf{R}^n$ such that $(1+\epsilon^{1/2^n} q(n))^{-1} r(B_1, B_2) \subset D \subset (1+\epsilon^{1/2n} q(n)) r(B_1, B_2)$.*

Proof: The statement of the proposition is invariant under scaling, so it is enough to show the proposition for quasisymplectic 1-balls. For this we proceed by induction on $n = \dim \mathbf{R}^{2n}/2$.

The case $n = 1$ is immediate with $q(1) = 1$. Thus assume that the proposition is known for $n - 1 \geq 1$ and let $\beta: D(1) \to \mathbf{R}^{2n}$ be a $(1+\epsilon)$-quasisymplectic 1-ball where $\epsilon \leq 2^{-n}$. Write $D_1 = \beta_1(B(1)), D_2 = \beta_2(B(1))$.

Let $x_1 \in \partial B(1)$ be a point from the boundary of $B(1)$ and write $y_1 = \beta_1(x_1)$. Let $x_2 \in \partial B(1)$ be the unique point which satisfies $\varphi_0(x_1, x_2) = 1$ and write $y_2 = \beta_2(x_2)$. By our assumption we have $1 + \epsilon \geq \varphi_0(y_1, y_2) \geq 1 - \epsilon$.

Define $H_1 = \{z \in \mathbf{R}^n \mid \varphi_0(z, y_2) = 0\}, H_2 = \{z \in \mathbf{R}^n \mid \varphi_0(y_1, z) = 0\}$ and let $P_i: \mathbf{R}^n = \mathbf{R} \oplus H_i \to H_i$ be the canonical projection. Define $W_1 = \{x \in \mathbf{R}^n \mid \varphi_0(x, x_2) = 0\}, W_2 = \{x \in \mathbf{R}^n \mid \varphi_0(x_1, x) = 0\}$ and $\bar{\beta}_i = P_i \circ \beta_i \mid W_i$. Then $(\bar{\beta}_1, \bar{\beta}_2)$ is a map of the symplectic vector space $(W_1 \times W_2, \rho_0)$ into the symplectic vector space $(H_1 \times H_2, \rho_0)$. Moreover the restriction of ρ_0 to $W_1 \times W_2$ is the standard symplectic form on $\mathbf{R}^{2n-2}$, and the same is true for $\rho_0 \mid_{H_1 \times H_2}$.

We claim that $(\bar{\beta}_1, \bar{\beta}_2)$ is a $1 + \epsilon + 2\epsilon^2$-quasisymplectic 1-ball in $\mathbf{R}^{2n-2}$.

To see this let $z_1 \in W_1$ and write $\beta_1(z_1) = a_1 y_1 + \bar{\beta}_1(z_1)$. Then $\varphi_0(z_1, x_2) = 0$ and

$$|\varphi_0(\beta_1 z_1, \beta_2 x_2)| = |\varphi_0(a_1 y_1 + \bar{\beta}_1(z_1), y_2)| \geq |a_1|(1 - \epsilon).$$

Since β is an $(1+\epsilon)$-quasisymplectic 1-ball we conclude that $\mid a_1 \mid \leq \epsilon/1 - \epsilon$. The same argument applied to $z_2 \in W_2$ shows that $\beta_2(z_2) = a_2 y_2 + \bar{\beta}_2(z_2)$ with $\mid a_2 \mid \leq \epsilon/1 - \epsilon$. Then

$$\mid \varphi_0(\bar{\beta}_1(z_1), \bar{\beta}_2(z_2)) - \varphi_0(\beta_1(z_1), \beta_2(z_2)) \mid = \mid a_1 a_2 \varphi_0(y_1, y_2) \mid \leq \epsilon^2 (1+\epsilon)/(1-\epsilon)^2$$

and hence

$$\mid \varphi_0(z_1, z_2) - \varphi_0(\bar{\beta}_1(z_1), \bar{\beta}_2(z_2)) \mid \leq \epsilon + \epsilon^2(1+\epsilon)/(1-\epsilon)^2 \leq \epsilon + 2\epsilon^2 \leq 2\epsilon.$$

We apply now the induction hypothesis to $(\bar{\beta}_1, \bar{\beta}_2)$. This means that there is a linear isomorphism $L_1: W_1 \equiv \mathbf{R}^{n-1} \to H_1 \equiv \mathbf{R}^{n-1}$ such that

$$\begin{aligned}&(1 + (2\epsilon)^{1/2^{n-1}} q(n-1))^{-1} (L_1 B(1), (L_1^t)^{-1} B(1)) \subset (\bar{\beta}_1 B(1), \bar{\beta}_2 B(1))\\ &\subset (1 + (2\epsilon)^{1/2^{n-1}} q(n-1))(L_1 B(1), (L_1^t)^{-1} B(1)).\end{aligned}$$

Define a linear map $L: \mathbf{R}^n \to \mathbf{R}^n$ by $L \mid_{W_1} = L_1$ and $L x_1 = y_1$.

Let $x = a_1 x_1 + z_1 \in \mathbf{R} x_1 \oplus W_1 \cap B(1)$ and write $\beta_1(x) = b_1 y_1 + \bar{z}_1 \in \mathbf{R} y_1 \oplus H_1$. Then we have $\max\{\varphi_0(x, w) \mid w \in W_2 \cap B(1)\} = \|z_1\| \leq 1$ and $\mid \varphi_0(x, w) -$

$\varphi_0(\beta_1 x, \bar\beta_2 w) \mid\leq 2\epsilon$ for all $w \in W_2 \cap B(1)$. By induction hypothesis, $\bar\beta_2(W_2 \cap B(1)) \supset (1 + 2\epsilon^{1/2^{n-1}} q(n-1))^{-1}(L_1^t)^{-1}B(1)$ and from this we conclude that $\bar z_1 = P_1\beta_1(x) \in (\|z_1\| + 2\epsilon)(1 + 2\epsilon^{1/2^{n-1}} q(n-1))L_1B(1) - (\|z_1\| - 2\epsilon)(1 + 2\epsilon^{1/2^{n-1}} q(n-1))^{-1}L_1B(1)$.

Similarly, $a_1 = \varphi_0(x, x_2)$ and $\mid \varphi_0(x, x_2) - \varphi_0(\beta_1 x, \beta_2 x_2) \mid\leq \epsilon$ implies that

$$\mid b_1 \mid= \varphi_0(\beta_1 x, \beta_2 x_2) \in [a_1 - \epsilon, a_1 + \epsilon].$$

Therefore, if $(\| \|_L)$ is the norm of the image of the euclidean scalar product under L, then

$$\|\beta_1 x\|_L^2 = b_1^2 + \|\bar z_1\|_L^2 \leq (a_1+\epsilon)^2 + (\|z_1\|_L + 2\epsilon)^2(1+2\epsilon^{1/2^{n-1}} q(n-1))^2 \leq a_1^2 + \|z_1\|^2 + \epsilon^{1/2^{n-1}} p$$

where $p \geq 1$ only depends on n. Together we conclude that $\|\beta_1 x\|_L \leq 1 + \epsilon^{1/2^n} \cdot q(n)$ where again $q(n) \geq 1$ is a universal constant.

In other words, $\beta_1 B(1) \subset (1 + \epsilon^{1/2^n} q(n))B_1$ where B_1 is the unit ball for $(\| \|_L)$. The same argument for β_2 then shows that $(D_1, D_2) \subset (1 + \epsilon^{1/2^n} q(n))(B_1, B_2)$.

This is the first half of our claim. The second half follows from the same argument by taking into account that $\|\beta_1 x\|_L^2 \geq (1 + \epsilon^{1/2^{n-1}} q(n))^{-1}$ for all $x \in \partial B(1)$. **q.e.d.**

From Proposition 3.2 the following is immediate:

Corollary 3.5: *For $\epsilon \leq 2^{-n}$ the volume of every $(1+\epsilon)$-quasisymplectic r-ball in $\mathbf{R}^n$ is contained in the interval* $[(1 + \epsilon^{1/2^n} q(n))^{-2n} r^{2n} a(n)^2, (1 + \epsilon^{1/2^n} q(n))^{2n} r^{2n} a(n)^2]$.

Let now again $\partial\tilde M$ be the ideal boundary of the universal convering of a negatively curved manifold M of dimension n. Denote by $[\ ]$ the logarithm of the cross ratio of the length cocycle of the metric on M.

Using the fact that $\partial\tilde M \times \partial\tilde M - \Delta$ is a smooth symplectic manifold and that for every $\xi \in \partial\tilde M$ the subsets $\partial\tilde M \times \{\xi\}$ and $\{\xi\} \times \partial\tilde M$ are Lagrangian we can define for every fixed $(\xi, \eta) \in \partial\tilde M \times \partial\tilde M - \Delta$ a function φ as in Lemma 3.1. We are going to see that this function is in fact given by the cross ratio $[\ ]$. First, for a compact set $K \subset \partial\tilde M$ with dense interior and non-empty complement define a function ρ_K on $(\partial\tilde M - K)^2$ by $\rho_K(\xi, \eta) = \sup\{[\xi, \eta, \zeta, \bar\zeta] \mid \zeta, \bar\zeta \in K\}$.

Then we have:

Lemma 3.6: *ρ_K is a distance on $\partial\tilde M - K$.*

Proof: First of all we clearly have $\rho_K(\xi, \xi) = 0$ for all $\xi \in \partial\tilde M - K$. Second, since $[\xi, \eta, \zeta, \bar\zeta] = -[\eta, \xi, \zeta, \bar\zeta] = [\eta, \xi, \bar\zeta, \zeta]$ the function ρ_K is symmetric and non-negative.

To show the triangle inequality let $\xi, \eta \in \partial\tilde M - K$ and let $\zeta, \bar\zeta \in K$ be such that $[\xi, \eta, \zeta, \bar\zeta] = \rho_K(\xi, \eta)$; such points exist by continuity of $[\ ]$ and compactness of K. If $\omega \in \partial\tilde M - K$ is arbitrary, then $\rho_K(\xi, \eta) = [\xi, \omega, \zeta, \bar\zeta] + [\omega, \eta, \zeta, \bar\zeta] \leq \rho_K(\xi, \omega) + \rho_K(\omega, \eta)$.

We are left with showing that $\rho_K(\xi, \eta) > 0$ for $\xi \neq \eta$; for this we need the fact that the interior U of K is non-empty. Choose $\zeta \in U$, let γ be a geodesic joining $\gamma(-\infty) = \zeta$ to $\gamma(\infty) = \xi$, and let $\bar\gamma$ be a geodesic joining $\bar\gamma(-\infty) = \zeta$ to $\bar\gamma(\infty) = \eta$. We assume that

$\gamma, \bar\gamma$ are parametrized in such a way that for every $t \in \mathbf{R}, \gamma'(t)$ and $\bar\gamma'(t)$ lie on the same strong unstable manifold. Corollary 2.10 of [H3] then shows that there is $\bar\zeta \in U \subset K$ in an arbitrarily small neighborhood of ζ such that $[\xi, \eta, \zeta, \bar\zeta] > 0$. From this the lemma follows. **q.e.d.**

Recall now from [H3] that $\partial\tilde M$ admits a C^1-structure if there is a differentiable structure for $\partial\tilde M$ for which π is a C^1-submersion. Then for every $v \in T^1\tilde M$ the restriction to $W^{su}(v)$ of the natural projection $\pi\colon T^1\tilde M \to \partial\tilde M$ is a C^1-diffeomorphism onto $\partial\tilde M - \pi(-v)$. In particular we can equip $\partial\tilde M$ with a continuous Riemannian metric, and every two such metrics are globally bilipschitz-equivalent. If ρ is any distance function on $\partial\tilde M$ which is locally bilipschitz equivalent to one (and hence every) continuous Riemannian metric on $\partial\tilde M$ we say that ρ is *locally equivalent* to a Riemannian structure. Then we have:

Proposition 3.7: *If $\partial\tilde M$ admits a C^1-structure then the distance functions ρ_K on $\partial\tilde M - K$ are locally equivalent to a Riemannian structure.*

Proof: Let $\emptyset \neq U \subset \partial\tilde M$ be open and connected with closure K and let $\zeta \in U$ be arbitrarily fixed. If $\eta, \xi \in \partial\tilde M - K = \Omega$ and if $\zeta_1, \zeta_2 \in K$ are such that $\rho_K(\xi,\eta) = [\xi,\eta,\zeta_1,\zeta_2]$ then $[\xi,\eta,\zeta_1,\zeta_2] = [\xi,\eta,\zeta_1,\zeta] + [\xi,\eta,\zeta,\zeta_2]$ shows that $\rho_K(\xi,\eta) \le 2\sup\{|[\xi,\eta,\zeta,\bar\zeta]| \mid \bar\zeta \in K\} = 2\bar\rho(\xi,\eta)$.

Thus it suffices to show that the function $\bar\rho$ which is defined by the above equation is locally equivalent to a Riemannian structure.

For this let $\xi \in \Omega$ again be arbitrary and let γ be a geodesic joining $\gamma(-\infty) = \zeta$ to $\gamma(\infty) = \xi$. Let d^{ss} (or d^{su}) be the distances on the leaves of W^{ss} (or W^{su}) induced by the restriction of the Sasaki metric and assume that γ is parametrized in such a way that for $v(\xi) = v = \gamma'(0)$ we have $\sup\{d^{ss}(v,w) \mid \pi(-w) \in K\} = 1$. Let moreover $r(\xi) > 0$ be the maximum of all numbers $r > 0$ such that $\{w \mid \pi(-w) \in K\}$ contains the ball of radius r about v in $W^{ss}(v)$. Notice that $\xi \to v(\xi)$ and $\xi \to r(\xi)$ are continuous.

Let $t \to v(t)$ be a curve of class C^1 in $W^{su}(v)$ through $v(0) = v$. For $w \in W^{ss}(v)$ choose a d^{ss}-geodesic φ_w joining $\varphi_w(0) = v$ to $\varphi_w(1) = w$. For $t \ge 0$ let $Y(t) \in T_{\varphi_w(t)}W^{su}$ be such that $d\pi(Y(t)) = d\pi(v'(0))$. Then $t \to Y(t)$ is uniformly continuous and

$$\frac{d}{dt}[\zeta, \pi(-w), \xi, \pi(v(t))]\,|_{t=0} = \int_0^1 d\omega(\varphi_w'(s), Y(s))\,ds$$

(see [H2]). In other words, if $\chi = \max\{|\int_0^1 d\omega(\varphi_w'(s), Y(s))\,ds| \mid \pi(-w) \in K\} < \infty$ then $\bar\rho(\xi, \pi v(t)) = t\chi + o(t)$ and hence $\bar\rho$ is locally Lipschitz at ξ with respect to the projection of $d^{su}\,|_{W^{su}(v)}$ to $\partial\tilde M$. From continuity and the considerations in [H2] and [H3] the proposition now immediately follows. **q.e.d.**

Remark : Proposition 3.7 shows in particular that $\partial\tilde M$ admits a C^1-structure only if the distances ρ_U ($\emptyset \neq U \subset \partial\tilde M$ open) are locally equivalent on the intersections of their domain of definition. Since the distances ρ_U are defined just by the length cocycle of the metric this shows that obstructions to existence of a C^1-structure on the geometric boundary $\partial\tilde M$ of M equipped with a fixed metric g can be read off immediately from the

cohomology class of the length cocycle. It seems to be reasonable to believe that local equivalence of the distances ρ_U implies the existence of a Lipschitz structure on $\partial\tilde{M}$.

Moreover, if $\partial\tilde{M}$ has a C^1-structure, then for every C^1-curve $c: S^1 \to \partial\tilde{M}$ the restriction of $[\,]$ to quadruples of pairwise distinct points in $c(S^1)$ is a signed Borel measure, i.e. is $\sigma-$additive and finite. It is not hard to see that rectifiable curves in $\partial\tilde{M}$ with respect to the C^1-structure are characterized by this property.

We apply now the above considerations to maps of $(\mathbf{R}^{2n-2}, \rho_0)$ into $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ equipped with the cross-ratio $[\,]$, where as before $\partial\tilde{M}$ is the ideal boundary of the universal covering of a compact negatively curved manifold M of dimension n.

For $c \geq 1$ define a *c-quasisymplectic map* of $D(r)$ into $\partial\tilde{M}\times\partial\tilde{M}-\Delta$ to be a continuous map β of $D(r)$ onto a compact subset of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ with the following properties:

1) There are continuous maps $\beta_i: B(r) \to \partial\tilde{M}(i = 1, 2)$ such that $\beta = (\beta_1, \beta_2)$.
2) $\mid [\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)] - \varphi_0(x, y) \mid \leq (c - 1)r^2$ for all $x \in B(r), y \in B(r)$.

Corollary 2.10 of [H3] then shows that for every $\epsilon > 0$ there is a number $r > 0$ such that for every $\delta \leq r$ and every $\xi \neq \eta \in \partial\tilde{M}$ there is a $1 + \epsilon$-quasisymplectic map β of $D(\delta)$ into $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ such that $\beta(0, 0) = (\xi, \eta)$.

We call the image of $D(r)$ under a $(1 + \epsilon)$-quasisymplectic map β a $(1 + \epsilon)$ *quasisymplectic r-ball* and call $\beta(0)$ the *center* of the quasisymplectic ball.

For $\epsilon > 0$ denote by $Q(\epsilon)$ the family of all $(1+\epsilon)-$quasisymplectic r-balls for arbitrary $r > 0$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. Then $Q(\epsilon) \subset Q(\delta)$ if $\epsilon \leq \delta$. For a $(1 + \epsilon)$-quasisymplectic r-ball $\beta: D(r) \to \partial\tilde{M} \times \partial\tilde{M} - \Delta$ define

$$\delta(\beta D(r)) = \sup\{|[\xi_1, q_1, \xi_2, q_2]| \mid (\xi_1, \xi_2) \in \beta(D(r)), (q_1, q_2) = \beta(0)\}.$$

Clearly $\delta(\beta D(r)) \in [(1 - \epsilon)r^2, (1 + \epsilon)r^2]$.

Fix any distance d on $\partial\tilde{M}$ which induces the natural topology and let $\mathrm{diam}(B)$ be the $d \times d$-diameter of a set $B \subset \partial\tilde{M} \times \partial\tilde{M}$. For $\epsilon > 0$ and a Borel-subset A of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ define

$$\mathcal{S}_\epsilon(A) = \inf\{\sum_{i=1}^{\infty} \delta(Q_i)^{n-1} a(n-1)^2 \mid Q_i \in Q(\epsilon), \mathrm{diam}(Q_i) \leq \epsilon, A \subset \cup_{i=1}^{\infty} Q_i\}$$

and let $\mathcal{S}(A) = \limsup_{\epsilon\to\infty} \mathcal{S}_\epsilon(A)$.

Denote by $\lambda = \lambda_g$ the Lebesgue-Liouville measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ induced by g. Then we have:

Proposition 3.8: $\mathcal{S} \leq \lambda$.

Proof: Recall that for every $v \in T^1\tilde{M}$ the restriction to $W^{su}(v)$ and $-W^{ss}(v)$ of the natural projection $\pi: T^1\tilde{M} \to \partial\tilde{M}$ is a homeomorphism onto $\partial\tilde{M} - \pi(-v)$ and $\partial\tilde{M} - \pi(v)$. Let g^{ss} be the Riemannian metric on TW^{ss} which is induced by the Riemannian metric on M. Let ω be the canonical contact form on T^1M and $T^1\tilde{M}$ and recall that the bundles TW^{ss}, TW^{su} are Lagrangian with respect to the symplectic form $d\omega$. Thus $d\omega$ determines an isomorphism of TW^{su} onto the dual $(TW^{ss})^*$ of TW^{ss} and hence the metric g^{ss} induces a metric g^{su} on TW^{su}. In fact, for $p \in \tilde{M}$ and for $v \in T^1_p\tilde{M}$ the fibre T_vW^{su} of TW^{su}

at v can be naturally identified with the g-orthogonal complement $v^\perp$ of v in $T_p\tilde{M}$. Then $v^\perp$ has also an identification with the tangent space T_v^v at v of the sphere $T_p^1\tilde{M}$. From the explicit form of $d\omega$ (see [H3]) we see that with these identifications the metric g^{su} on T_vW^{su} is just the natural metric on T_v^v, viewed as the tangent space of the unit sphere $T_p^1\tilde{M}$.

Clearly g^{su} is uniformly Hölder continuous, but its restriction to the leaves of W^{su} is not smooth.

However, for each $v \in T^1\tilde{M}$ we can define smooth coordinates $\varphi_v : B \to W^{su}(v)$ for $W^{su}(v)$ at v where B is the unit ball in $\mathbf{R}^{n-1}$ with the following properties:

1) $\varphi_v(0) = v$ and $d\varphi_v(0) : T_0\mathbf{R}^{n-1} \to (T_vW^{su}, g^{su})$ is an isometry.
2) If $\varphi_v^* g^{su}$ denotes the pull-back of g^{su} under φ_v, then $(x, v) \in B \times T^1\tilde{M} \to \varphi_v^* g^{su}(x)$ is uniformly Hölder continuous.

Let moreover $\exp_v : T_vW^{ss} \to W^{ss}(v)$ be the exponential map of the Riemanian metric g^{ss} and denote by $\|\|$ the norm of TW^i induced by $g^i (i = ss, su)$.

For $v \in T^1\tilde{M}$ define a map

$$E_v : T_vW^{ss} \oplus T_vW^{su} \to \partial\tilde{M} \times \partial\tilde{M}$$

by $E_v(X, Y) = (\pi(-\exp_v X), \pi(\varphi_v(Y)))$ (here we identify T_vW^{su} and $\mathbf{R}^{n-1}$ via $d\varphi_v(0)$). Then E_v is a homeomorphism of an open neighborhood U of 0 in $T_vW^{ss} \oplus T_vW^{su}$ onto a neighborhood of $(\pi(-v), \pi(v))$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ which is absolutely continuous with respect to the volume element on $T_vW^{ss} \oplus T_vW^{su}$ induced by the symplectic form $d\omega$ and the Lebesgue-Liouville current λ, with locally uniformly Hölder continuous Jacobian whose value at $(0, 0) \in T_vW^{ss} \oplus T_vW^{su}$ equals 1.

By the definition of the map E_v there is for every fixed $\epsilon > 0$ a number $\delta = \delta(\epsilon) > 0$ such that for every $v \in T^1\tilde{M}$, every $X_i \in T_vW^i$ with $\|X_i\| \leq \delta (i = ss, su)$ the following is satisfied (this is Corollary 2.10 of [H3]):

i) The Jacobian of E_v at $X_{ss} + X_{su}$ is contained in $[1 - \epsilon, 1 + \epsilon]$.
ii) $|[\pi(-\exp_v X_{ss}), \pi(-v), \pi(\varphi_v X_{su}), \pi(v)] - d\omega(X_{ss}, X_{su})| \leq \epsilon\delta^2$.

Let now $\delta \leq \delta(\epsilon)$, let $v \in T^1\tilde{M}$ and let B^i be a ball of radius $r < \delta$ in (T_vW^i, g^i). By i) above we then have $\lambda(\pi(-\exp_v B^{ss}), \pi\varphi_v B^{su}) \in a(n-1)^2 r^{2n-2}[1 - \epsilon, 1 + \epsilon]$, and ii) shows that the restriction of E_v to $B^{ss} \times B^{su}$ is a $(1+\epsilon)$-quasisymplectic r-ball. Now if $A \subset \partial\tilde{M} \times \partial\tilde{M} - \Delta$ is any compact set, then for every $\epsilon > 0$ there are points $v_1, \dots v_k \in T^1\tilde{M}$ and balls $B_j^i \subset T_{v_j}W^i (i = ss, su)$ of radius $r_j \leq \delta(\epsilon)$ such that

$$A \subset \cup_{j=1}^k E_{v_j}(B_j^{ss} \times B_j^{su}) \quad \text{and} \quad \sum_{j=1}^k \lambda(E_{v_j}(B_j^{ss} \times B_j^{su})) \leq \lambda(A) + \epsilon.$$

But this means $\lambda(A) \geq (1 - \epsilon)\sum_{j=1}^k a(n-1)^2 r_j^{2n-2} - \epsilon$, and consequently $\mathcal{S}_\epsilon(A) \leq \lambda(A) + \epsilon/1 - \epsilon$.

Since $\epsilon > 0$ was arbitrary and $\mathcal{S}_\delta \geq \mathcal{S}_\epsilon$ for $\delta \leq \epsilon$ we conclude that $\mathcal{S}_\epsilon(A) \leq \lambda(A)$ for all $\epsilon > 0$ and therefore $\mathcal{S} \leq \lambda$ as claimed. **q.e.d.**

We assume now for the moment that the Anosov splitting of TT^1M is of class C^1. Our goal is to study more precisely the measure $\mathcal{S}$..

For this recall the definition of the *Kanai connection* ∇ on T^1M which is defined as follows: Let $\mathcal{J}$ be the $(1,1)$-tensor field on T^1M defined by $\mathcal{J}(Y) = -Y$ for $Y \in TW^{ss}$, $\mathcal{J}Y = Y$ for $Y \in TW^{su}$ and $\mathcal{J}(\mathbf{R}X^0) = 0$. The indefinite metric h on T^1M defined by $h(Y,Z) = d\omega(Y, \mathcal{J}Z) + \omega \oplus \omega(Y,Z)$ is of class C^1 and hence there is a unique affine connection ∇ on T^1M such that

i) h is parallel with respect to ∇

ii) The torsion of ∇ is $d\omega \otimes X^0$.

The bundles TW^{su}, TW^{ss} are invariant under ∇, and the restriction of ∇ to every leaf of W^{su} or W^{ss} is flat. In other words, if we denote the lift of ∇ to $T^1\tilde{M}$ by the same symbol, then for every $v \in T^1\tilde{M}$ there is a basis of $TW^{su} \mid W^{su}(v)$ consisting of ∇-parallel vector fields.

The space of geodesics in $\tilde{M}$ is $\partial\tilde{M} \times \partial\tilde{M} - \Delta = T^1\tilde{M}/\mathbf{R}$ where $\mathbf{R}$ acts on $T^1\tilde{M}$ as the geodesic flow. Equipped with the projection of $d\omega$ this is a smooth symplectic manifold.

According to Darboux's theorem, every point $\xi \in \partial\tilde{M} \times \partial\tilde{M} - \Delta$ admits a neighborhood U and local coordinates $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ such that in these coordinates the symplectic form $d\omega$ is the standard symplectic form $\sum dx_i \wedge dy_i$ on $\mathbf{R}^{2n-2}$.

We want to use the Kanai-connection ∇ to construct particular such coordinates which are adapted to the product structure of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

For this let $v \in T^1\tilde{M}$ be arbitrary and let $\exp_v^{su}: T_vW^{su} \to W^{su}(v)$ and $\exp_v^{ss}: T_vW^{ss}(v) \to W^{ss}(v)$ be the exponential map of the restriction of ∇ to $W^{su}(v)$ and $W^{ss}(v)$ respectively. Choose open neighborhoods A^{su} of v in $W^{su}(v)$, A^{ss} of v in $W^{ss}(v)$ such that for every $w \in A^{su}$ and $z \in A^{ss}$ the intersection $W^{ss}(w) \cap W^u(z)$ consists of a unique point $[w,z]$. Then $H = \{[w,z] \mid w \in A^{su}, z \in A^{ss}\}$ is a local hypersurface in $T^1\tilde{M}$ of class C^1 transversal to the geodesic flow and hence $(H, d\omega)$ is a symplectic manifold which can naturally be identified with a neighborhood of $(\pi(v), \pi(-v))$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. For each point $z \in H$ the intersection $W^{ss}(z) \cap H$ is an open neighborhood of z in $W^{ss}(z)$. Let $X_1, \dots, X_{n-1}$ be a basis of T_vW^{su}, $Y_1, \dots, Y_{n-1}$ be the basis of T_vW^{ss} which is dual with respect to $d\omega$ (i.e. such that $d\omega(X_i, Y_j) = \delta_{ij}$) and define local coordinates $\Psi: \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \to H$ by $\Psi(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}) = [\exp_v^{su}(\sum x_i X_i), \exp_v^{ss}(\sum y_j Y_j)]$.

Via these coordinates the symplectic structure on H induces a function φ as in Lemma 3.1.

Lemma 3.9: *Let $x \in A^{su}, y \in A^{ss}$. Then $[\pi(x), \pi(v), \pi(-y), \pi(-v)] = \varphi(\Psi^{-1}[x,y])$.*

Proof: Let $\beta: [0,1]^2 \to H$ be a map of class C^1 with $\beta(0,0) = v$, $\beta(s,0) \in A^{su}$, $\beta(0,t) \in A^{ss}$ and $\beta(s,t) = [\beta(s,0), \beta(0,t)]$. For each fixed $s \in [0,1]$ the curve $t \to \beta(s,t)$ is contained in a leaf of W^{ss}, and $d\pi(\frac{\partial}{\partial s}\beta(s,t))$ is independent of $t \in [0,1]$. In other words, $\frac{\partial}{\partial s}\beta$ is parallel along the curves $t \to \beta(s,t)$.

Lemma 1 of [H2] then shows that

$$\frac{d}{ds}[\pi\beta(s,0),\pi(v),\pi(-\beta(0,1)),\pi(-v)] =$$
$$\frac{d}{ds}([\pi\beta(s,0),\pi(v),\pi(-\beta(0,1)),\pi(-\beta(s,1))]+$$
$$[\pi\beta(s,0),\pi(v),\pi(-\beta(s,1)),\pi(-v)]) =$$
$$\int_0^1 d\omega(\frac{\partial}{\partial s}\beta(s,t),\frac{\partial}{\partial t}\beta(s,t))\,dt$$

since $\pi(-\beta(0,1)) = \pi(-\beta(s,1))$ for all s. From this the lemma is immediate. **q.e.d.**

On the other hand, X_i, Y_j can be extended to ∇-parallel vector fields along $W^{su}(v)$ which we denote by the same symbol. Define local coordinates $\Psi_0 \colon \mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\to H$ by

$$\Psi_0(x_1,\dots,x_{n-1},y_1,\dots y_{n-1}) =$$
$$\exp^{ss}(\sum_j y_jY_j(\exp_v^{su}(\sum x_iX_i))).$$

Lemma 3.10: $d\omega = \sum_{i=1}^{n-1} dx_i\wedge dy_i$ *in the coordinates* Ψ_0.

Proof: Let $c\colon(-\epsilon,\epsilon)\to W^{su}(v)$ be the integral curve of a ∇-parallel vector field along $W^{su}(v)$, and for some ∇-parallel section Y of TW^{ss} over $W^{su}(v)$ let $\beta(s,t) = \exp^{ss}tY(c(s))$. We want to show that $d\omega(\frac{\partial}{\partial s}\beta,\frac{\partial}{\partial t}\beta)$ is independent of s and t. Since $d\omega$ is ∇-parallel we conclude that this function is constant along the curve $s\to\beta(s,0)$. On the other hand, for every fixed $s\in(-\epsilon,\epsilon)$ we have $\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\beta = 0$ and consequently (formally)

$$\frac{\partial}{\partial t}d\omega(\frac{\partial}{\partial s}\beta,\frac{\partial}{\partial t}\beta) = d\omega(\frac{\nabla}{\partial t}\frac{\partial}{\partial s}\beta,\frac{\partial}{\partial t}\beta) =$$
$$d\omega(\frac{\nabla}{\partial s}\frac{\partial}{\partial t}\beta,\frac{\partial}{\partial t}\beta) = \frac{1}{2}\frac{\partial}{\partial s}d\omega(\frac{\partial}{\partial t}\beta,\frac{\partial}{\partial t}\beta) = 0$$

(recall that the torsion of ∇ equals $d\omega\otimes X^0$). From this the lemma is immediate. **q.e.d.**

In the sequel we call coordinates $(x_1,\dots,x_{n-1},y_1,\dots y_{n-1})$ for $\partial\tilde M\times\partial\tilde M-\Delta$ of class C^1 with the property that the symplectic form $d\omega$ has the standard form $d\omega=\sum dx_i\wedge dy_i$ *Darboux-coordinates*. The lemma of Darboux says that Darboux coordinates exist always. The Darboux coordinates Ψ_0 on the symplectic manifold H (which we can canonically identify with an open subset of $\partial\tilde M\times\partial\tilde M-\Delta$) are particularly well adapted to the product structure of $\partial\tilde M\times\partial\tilde M-\Delta$. Namely, for every fixed $y\in\mathbf{R}^{n-1}$ the set $\Psi_0(\{y\}\times\mathbf{R}^{n-1})$ is contained in a leaf of the strong stable foliation. However, it is not true that $\Psi_0(\mathbf{R}^{n-1}\times\{y\})$ is contained in a leaf of the unstable foliation. For otherwise the manifold $\partial\tilde M\times\partial\tilde M-\Delta$ equipped with the Kanai connection would be flat which is impossible.

Darboux coordinates with the above additonal properties are by no means unique. We can choose them compatible with a symplectic submanifold of $\partial\tilde M\times\partial\tilde M-\Delta$ which is a product.

More precisely, let $v \in T^1\tilde{M}$ and let $A \subset W^{su}(v), B \subset W^{ss}(v)$ be k-dimensional embedded submanifolds containing v in their interior. Assume that the restriction of $d\omega$ to $L = \{[x,y] \mid x \in A, y \in B\}$ is non-degenerate. Let $X_1, \dots, X_{n-1}$ be a basis of T_vW^{su} such that $X_1, \dots, X_k$ is a basis of T_vA and that $X_{k+1}, \dots, X_{n-1}$ is a basis of the annihilator of T_vB in the dual of T_vW^{ss}, where we identify this dual space with T_vW^{su} via $d\omega$. Let $Y_1, \dots, Y_{n-1}$ be the basis of T_vW^{ss} which is dual to the basis $X_1, \dots, X_{n-1}$; then $Y_1, \dots, Y_k$ is basis of T_vB. Extend $Y_1, \dots, Y_{n-1}$ by parallel transport to vector fields on T_vW^{su}. For every $x \in A$ and every $j \le k$ the vector $Y_j(x)$ is tangent to L and hence $Y_1, \dots, Y_k \mid A$ defines a trivialization of the dual of TA which induces a trivialization of TA. Similarly, by the dual procedure, for each $x \in A$ we obtain a trivialization of $[x, B]$. Using the exponential map of these trivializations we obtain as before Darboux coordinates for L which can be extended to Darboux coordinates for H (at least in a neighborhood of v).

Lemma 3.11: *If $\partial\tilde{M}$ has a C^1-structure, then for every $\epsilon > 0$ there is a number $\chi = \chi(\epsilon) > 0$ with the following property: For every $r > 0$ and every $(1+\chi)$-quasisymplectic map β with diam $\beta(D(r)) < \chi$ there is a symplectic map $\beta_0: D(r(1+\epsilon)) \to \partial\tilde{M} \times \partial\tilde{M} - \Delta$ such that $\beta_0 D(r(1-\epsilon)) \subset \beta D(r) \subset \beta_0 D(r(1+\epsilon))$.*

Proof: The statement of the lemma is local and will be proved by a local argument.

Let ρ be a continuous symplectic form on $\mathbf{R}^n \times \mathbf{R}^n$ with the following properties:

1) At every point in $\{0\} \times \mathbf{R}^n \cup \mathbf{R}^n \times \{0\}, \rho$ coincides with the standard symplectic form $\rho_0 = \sum dx_i \wedge dy_i$.
2) For every $x \in \mathbf{R}^n$ the submanifolds $\{x\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{x\}$ are Lagrangian with respect to ρ, in particular we can define the function φ as in the beginning of this section.
3) For every point $0 \neq x \in \mathbf{R}^n$ with sufficiently small euclidean norm the sets $A(x) = \{y \in \mathbf{R}^n \mid \varphi(x,y) = 0\}$ and $C(x) = \{y \in \mathbf{R}^n \mid \varphi(y,x) = 0\}$ are C^1- hypersurfaces in $\mathbf{R}^n$ whose tangent spaces at 0 coincide with the tangent space at 0 of the hypersurface $A_0(x) = \{y \mid \varphi_0(x,y) = 0\}$ and $C_0(x) = \{y \in \mathbf{R}^n \mid \varphi_0(y,x) = 0\}$ where as before, φ_0 is defined by the standard form ρ_0.
4) For a fixed compact ball B about 0 in $\mathbf{R}^n$ and a unit vector $x \in \mathbf{R}^n$ the hypersurfaces $A(tx) \cap B$ and $C(tx) \cap B$ converge as $t \to 0$ in the C^1-topology to $A_0(x)$.
5) For every $\epsilon > 0$ there is a number $\chi = \chi(\epsilon) > 0$ such that $|\varphi(x,y)| \le (1+\epsilon) \parallel x \parallel \parallel y \parallel$ and $\varphi(sx, tx) \ge (1+\epsilon)^{-1} st\|x\|^2$ for all $x, y \in \mathbf{R}^n$ with $\|x\| < \chi, \|y\| < \chi$ and $s, t \in [0,1]$.

Using the explicit form of the local coordinates on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ given in Lemma 3.9 we see that the above properties are satisfied for the function φ defined by the cross ratio near a given point.

Use the function φ to define a notion of a $(1+\epsilon)$-quasisymplectic r-ball on $\mathbf{R}^n \times \mathbf{R}^n$. We want to show by induction on n that the analogue of the statement of the lemma for this function φ is satisfied. The argument used is similar to the one given in the proof of Proposition 3.4.

The case $n = 1$ is obvious, so assume that the above claim is known for some $n-1 \ge 1$.

Let $\varphi: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be as above. By 3) above there is a number $\chi_1 > 0$ such that for every $0 \neq x \in \mathbf{R}^n$ with $\| x \| < \chi_1$ the euclidean angle between every $0 \neq y \in A(x) \cap B(\chi_1)$ and $A_0(x)$ is smaller than $\pi/16$.

Let $\kappa \in (0, \pi/16)$, let $\chi = \min\{\chi_1, \chi(\kappa)\}$ where $\chi(\kappa) > 0$ is as in 5) above, let $r > 0$ and let $\beta: D(r) \to B(\chi) \times B(\chi)$ be a $(1+\kappa)$-quasisymplectic embedding. By our definition we can write $\beta = (\beta_1, \beta_2)$ where β_i is a continuous map of $B(r)$ into $\mathbf{R}^n$ with $\beta_i(0) = 0$.

For every x from the boundary $\partial B(r)$ of $B(r)$ choose a point $f(x) \in \partial B(r)$ such that $\beta_2 f(x)$ is contained in the half-line through 0 spanned by $\beta_1(x)$. Such a point exists since $\beta_2 B(r)$ is a neighborhood of 0. Since the maps β_i are continuous we can arrange in such a way that $f(x)$ depends continuously on x.

We want to find some $x \in \partial B(r)$ such that the angle between x and $f(x)$ is not larger than $\pi/8$. For this choose a point $x \in \partial B(r)$ such that the euclidean norm $\nu_1 = \| \beta_1(x) \|$ of $\beta_1(x)$ is maximal. For $\nu_2 = \| \beta_2(fx) \|$ we then have $\varphi(\beta_1(x), \beta_2(fx))/\nu_1\nu_2 \geq (1+\kappa)^{-1}$.

By our assumption on a quasisymplectic r-ball there is a point $y \in \partial B(r)$ such that $\varphi(\beta_1(y), \beta_2(fx)) \geq r^2(1-\kappa)$. Since $\varphi(\beta_1(y), \beta_2(fx)) \leq (1+\kappa) \| \beta_1(y) \| \| \beta_2(fx) \|$ and $\|\beta_1(y) \| \leq \nu_1$ we conclude that $\nu_2 \geq r^2(1-\kappa)/(1+\kappa)\nu_1$ and consequently $\varphi(\beta_1 x, \beta_2(fx)) \geq r^2(1-\kappa)/(1+\kappa)^2$. Again by the definition of a quasisymplectic r-ball this implies that the angle between x and fx is not larger than $\pi/8$ and is close to zero if χ and κ are sufficiently close to 0.

Now let x be a point as above; then $C(\beta_2 fx) \times A(\beta_1 x)$ is a symplectic manifold, and the projection of $\beta \mid C_0(fx) \times A_0(x)$ to $C(\beta_2 fx) \times A(\beta_1 x)$ is a $(1+\kappa')$- quasisymplectic embedding where $\kappa' > \kappa$ depends on κ and χ and tends to zero with κ and χ. By our induction hypothesis we can find symplectic coordinates on $C(\beta_2 fx) \times A(\beta_1 x)$ such that with respect to these coordinates the projection of $\beta \mid C_0(fx) \times A_0(x)$ contains $D(r(1-\epsilon))$ and is contained in $D(r(1+\epsilon))$ where $\epsilon > 0$ depends on κ' and χ and tends to zero as $\kappa' \to 0$ and $\chi \to 0$. With the arguments in the proof of Proposition 3.4 and the explicit way to construct suitable symplectic coordinates described in Lemma 3.10 we can extend these symplectic coordinates to coordinates on a neighborhood of $\beta D(r)$ with the required properties. **q.e.d.**

As an immediate corollary to Proposition 3.8 and Lemma 3.11 we obtain:

Corollary 3.12: *If the Anosov splitting of TT^1M is of class C^1, then $\mathcal{S} = \lambda$.*

We can also use quasisymplectic balls in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ to define a packing measure as follows: For $\epsilon > 0$ and an open subset U of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ write

$$\mathcal{P}_\epsilon(U) = \sup\{\sum_{i=1}^{\infty} \delta(Q_i)^{n-1} a(n-1)^2 \mid Q_i \in Q(\epsilon), \mathrm{diam} Q_i \leq \epsilon, Q_i \cap Q_j = \emptyset\}$$

and let $\mathcal{P}(U) = \liminf_{\epsilon \to 0} \mathcal{P}_\epsilon(U)$. If $A \subset \partial\tilde{M} \times \partial\tilde{M} - \Delta$ is any Borel set, then we define $\mathcal{P}(A) = \inf\{\mathcal{P}(U) \mid U \supset A\}$. From the arguments in the proof of Proposition 3.8 and Lemma 3.11 we infer:

Proposition 3.13: *a)* $\lambda \leq \mathcal{P}$ *b) If the Anosov splitting of* TT^1M *is of class* C^1 *then* $\lambda = \mathcal{P}$.

Corollary 3.14: *Let* M, N *be compact negatively curved manifolds which have the same marked length spectrum. If the Anosov splitting of* TT^1M *is of class* C^1, *then* M *and* N *have the same volume.*

Remark: A more natural way to obtain a current from a positive Hölder class would be to drop in the definition of $\mathcal{S}$ the requirement that the covering family consists of quasisymplectic balls. More precisely, if we define

$$\bar{\mathcal{S}}_\epsilon(A) = \inf\{\sum_{i=1}^{\infty} a(n-1)^2\delta(A_i \times B_i)^{n-1} \mid A \subset \cup_{i=1}^{\infty} A_i \times B_i,\ \text{diam}\ (A_i \times B_i) \leq \epsilon\}$$

and $\bar{\mathcal{S}}(A) = \limsup_{\epsilon\to 0} \bar{\mathcal{S}}_\epsilon(A)$, then clearly $\bar{\mathcal{S}} \leq \mathcal{S}$. The Blaschke-Santaló inequality for $\mathbf{R}^n$ indicates that probably $\bar{\mathcal{S}} = \mathcal{S}$ always. The only case where I was able to check this equality is the case of a hyperbolic 3-manifold.

4. Intersection in higher dimensions

In this section we consider again a closed negatively curved manifold M. As before, we denote by $\partial\tilde{M}$ the ideal boundary of the universal covering $\tilde{M}$ of M.

The Riemannian metric g on M lifts to a Riemannian metric g^u on the leaves of W^u. This metric defines a family of Lebesgue measures λ^{su} on the leaves of W^{su}. The measures λ^{su} are quasi-invariant under the action of the geodesic flow and they transform via

$$\frac{d}{dt}\lambda^{su} \circ \Phi^t \mid_{t=0} = trU.$$

Here for every $v \in T^1M$, $trU(v)$ is just the trace of the second fundamental form at Pv of the horosphere $PW^{su}(v)$.

The function $v \to trU(v)$ is Hölder continuous, positive and of pressure zero and consequently it defines a positive Hölder class which does not depend on the particular choice of the family of smooth measures on strong unstable manifolds. Its Gibbs equilibrium state is just the Lebesgue Liouville measure λ on T^1M. We call the class defined by trU the *Lebesgue-class*.

Similarly, the image of λ^{su} under the flip $\mathcal{F}$ is a family λ^{ss} of Lebesgue measures on strong stable manifolds. If $U(v)$ denotes the second fundamental operator of the horosphere $PW^{su}(v)$, viewed as a linear automorphism of the orthogonal complement $v^\perp$ of v, then $f(v) = \det(U(v) + U(-v))$ is a positive Hölder continuous function on T^1M, and $\lambda = dt \times d\lambda^{su} \times fd\lambda^{ss}$ where dt is the 1-dimensional Lebesgue measure on the flow-lines of the geodesic flow (see [H3]).

The measure λ is invariant under the flip $\mathcal{F}: v \to -v$ and hence the cocycle ζ defined by trU is equivalent to the cocycle $\zeta \circ \mathcal{F}$ defined by $trU \circ \mathcal{F}$ where an equivalence is given by the function $\log f$.

Following [H5], we use the cocycle $\zeta \circ \mathcal{F}$ to define a Hölder continuous kernel k: $\tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbf{R}$. For every fixed $\xi \subset \partial\tilde{M}$ the function $(x, y) \in \tilde{M} \to k(x, y, \xi)$ is smooth. Moreover $k(x, x, \xi) = 0$, and if $v \in T^1_x\tilde{M}$ is such that $\pi(v) = \xi$, then there is a smooth function $\beta: W^{ss}(v) \to \mathbf{R}$ such that $k(x, \cdot, \xi)^{-1}(0) = \{P\Phi^{\beta(w)}w \mid w \in W^{ss}(v)\}$. From the kernel k in turn we obtain the Lebesgue cross ratio (see [H5]). For this fix a point $x \in \tilde{M}$ and for $v \neq w \in T^1_x\tilde{M}$ choose a geodesic γ in $\tilde{M}$ joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$ and write $\alpha(v, w) = \frac{1}{2}(\log f(\gamma'(0)) - k(\gamma(0), x, \pi(v)) - k(\gamma(0), x, \pi(w))$. Then $\alpha(v, w)$ does not depend on the choice of γ. Moreover by Lemma 1.2 of [H5] and the above considerations, for fixed $\xi \neq \eta \in \partial\tilde{M}$ the function $x \in \tilde{M} \to \alpha(\pi^{-1}(\xi) \cap T^1_x\tilde{M}, \pi^{-1}(\eta) \cap T^1_x\tilde{M})$ is smooth.

The *Lebesgue-cross ratio* is then the function [ ] on the space of quadruples of pairwise distinct points in $\partial\tilde{M}$ defined as follows: Choose $x \in \tilde{M}$, let $v_i \in T^1_x\tilde{M}$ be such that $\pi(v_i) = \xi_i$ and write $[\xi_1, \xi_2, \xi_3, \xi_4] = \alpha(v_1, v_3) + \alpha(v_2, v_4) - \alpha(v_1, v_4) - \alpha(v_2, v_3)$.

This function admits a continuous extension to the space of quadruples $(\xi_1, \xi_2, \xi_3, \xi_4)$ with $\xi_2 \neq \xi_3$ and $\xi_1 \neq \xi_4$ and hence can be viewed as a function on the space of pairs of oriented geodesics in $\tilde{M}$, where we identify $(\xi_1, \xi_2, \xi_3, \xi_4)$ with the pair (γ_1, γ_2) of geodesics with endpoints $\gamma_1(-\infty) = \xi_3, \gamma_1(\infty) = \xi_1, \gamma_2(-\infty) = \xi_4$ and $\gamma_2(\infty) = \xi_2$. Since the space $\mathcal{G}\tilde{M}$ of geodesics in $\tilde{M}$ is naturally a smooth manifold (via the identification $\mathcal{G}\tilde{M} = T^1\tilde{M}/\mathbf{R}$ where $\mathbf{R}$ acts as the geodesic flow), the cross ratio is therefore a function on a smooth manifold. Notice that the Lebesgue cross ratio is in general different for the cross ration which we used in Section 3.

In the next lemma we investigate the regularity of [ ].

Lemma 4.1: *If the Anosov splitting of TT^1M is of class C^k for some $k \geq 1$, then [ ] is of class C^k.*

Proof: If the Anosov splitting is of class C^k, then the functions trU and $v \to \det(U(v) + U(-v))$ are of class C^k and $\partial\tilde{M}$ admits a C^k-structure in the sense of [H3]. Then the kernel $k: \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbf{R}$ is of class C^k as well, and from this and the definition of [ ] the lemma easily follows. **q.e.d.**

The following proposition is an analogue of Corollary 2.13 of [H3]:

Proposition 4.2: *If [ ] is of class C^k for some $k \geq 1$, then $\partial\tilde{M}$ admits a C^k-structure.*

Proof: Define a cocycle $\beta(v, t)$ for the geodesic flow on T^1M and $T^1\tilde{M}$ by $\beta(v, \zeta \circ \mathcal{F}(v, t)) = t$. Then $\Psi^t(v) = \Phi^{\beta(v,t)}v$ defines a Hölder continuous times change for Φ^t which is smooth along the leaves of the stable foliation. The resulting flow is Anosov in the following sense:

1) For every $v \in T^1M$ the set $\bar{W}^{ss}(v) = \{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0 (t \to \infty)\}$ is a smoothly embedded submanifold of $W^s(v)$ which can be realized as the graph of a smooth function on $W^{ss}(v)$.

2) For every $v \in T^1M$ the set $\bar{W}^{su}(v) = \{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0 (t \to -\infty)\}$ is a Hölder submanifold of $W^u(v)$ which can be realized as the graph of a Hölder continuous function on $W^{su}(v)$.

Let now A_1, A_2 be open, relative compact subsets of $\partial\tilde{M}$ whose closures do not intersect. Write $\Omega = \{v \in T^1\tilde{M} \mid \pi(v) \in A_1, \pi(-v) \in A_2\}$. By 1) and 2) above, for all $v, w \in \Omega$ the intersection $\bar{W}^{ss}(v) \cap W^u(w)$ consists of a unique point $q(v,w)$. Let $\bar{\sigma}(v,w) \in \mathbf{R}$ be the unique number $\tau \in \mathbf{R}$ such that $q(v,w) \in \Psi^\tau \bar{W}^{su}(w)$. Since $q(v, \Psi^t w) = q(v,w)$ and $q(\Psi^t v, w) = \Psi^t q(v,w)$ we have $\bar{\sigma}(\Psi^t v, w) = \bar{\sigma}(v,w) + t, \bar{\sigma}(v, \Psi^t w) = \bar{\sigma}(v,w) - t$ and $\bar{\sigma}(v,w) + \bar{\sigma}(w,v) = [\pi(-v), \pi(-w), \pi(v), \pi(w)]$ (compare [H3]).

For a fixed point $v \in \Omega$ and $w_1 \in \bar{W}^{ss}(v), w_2 \in \bar{W}^{su}(v)$ we have $\bar{\sigma}(w_1, w_2) = 0$ and consequently $[\pi(-w_1), \pi(-w_2), \pi(w_1), \pi(w_2)] = \bar{\sigma}(w_2, w_1)$.

Assume from now on that the cross ratio is a function of class C^k on $\mathcal{G}\tilde{M} \times \mathcal{G}\tilde{M}$ for some $k \geq 1$.

We follow the proof of Corollary 2.13 of [H3] and use the cross ratio [ ] to construct C^k-coordinates for the manifolds $\bar{W}^{ss}$.

First, let $U \subset T^1\tilde{M}$ be a nontrivial open subset. For $v, w \in T^1\tilde{M}$ write $Cr(v,w) = \bar{\sigma}(v,w) + \bar{\sigma}(w,v) = [\pi(-v), \pi(-w), \pi(v), \pi(w)]$. We determine inductively points

$$w_1, \ldots, w_i \in T^1\tilde{M} (1 \leq i \leq n-1)$$

and nontrivial open sets $B_1 \supset \ldots \supset B_i$ of U such that for every $w \in B_i$ and every $j \leq i$ the following is satisfied:

i) The restriction of $f_j = Cr(\cdot, w_j)$ to $B_i \cap \bar{W}^{ss}(w)$ does not have critical points.

ii) $\cap_{j=1}^i f_j^{-1}(f_j(w)) \cap \bar{W}^{ss}(w) \cap B_i$ is a C^k-embedded connected submanifold of $\bar{W}^{ss}(w) \cap B_i$ of codimension i.

Let $0 \leq i \leq n-1$ and assume that B_i and $w_1, \ldots, w_i$ are already determined. Let $w \in B$ and let $c \colon (-\epsilon, \epsilon) \to \cap_{j=1}^i f_j^{-1}(f_j(w)) \cap \bar{W}^{ss}(w) \cap B_i$ be an embedded curve of class C^k through $c(0) = w$ with nowhere vanishing tangent. Write

$$\xi = \pi(w), \eta_1 = \pi(-w), \eta_2 = \pi(-c(\epsilon/2));$$

the points $\xi, \eta_1, \eta_2 \in \partial\tilde{M}$ are pairwise distinct. We claim that there are $\beta_1 \neq \beta_2 \in \partial\tilde{M} - \{\xi, \eta_1, \eta_2\}$ such that $[\eta_1, \beta_1, \beta_2, \xi] \neq [\eta_2, \beta_1, \beta_2, \xi]$. For otherwise we conclude by continuity that $[\eta_1, \eta_2, \beta, \xi] = 0$ for all $\beta \in \partial\tilde{M} - \{\eta_1, \eta_2, \xi\}$ and consequently also $[\eta_1, \eta_2, \beta_1, \beta_2] = [\eta_1, \eta_2, \beta_1, \xi] + [\eta_1, \eta_2, \xi, \beta_2] = 0$ for all β_1, β_2.

Since [ ] is continuous and $\pi_1(M)-$invariant and since the action of $\pi_1(M)$ on the space of geodesics in $\partial\tilde{M}$ is minimal this would mean that [ ] vanishes identically, a contradiction.

Choose $w_{i+1} \in T^1\tilde{M}$ in such a way that

$$[\eta_1, \pi(-w_{i+1}), \pi(w_{i+1}), \xi] \neq [\eta_2, \pi(-w_{i+1}), \pi(w_{i+1}), \xi].$$

Then $Cr(w, w_{i+1}) \neq Cr(c(\epsilon/2), w_{i+1})$. Since the function $f_{i+1} \colon z \to Cr(z, w_{i+1})$ is of class C^k it can be used as a coordinate function for the leaves of $\bar{W}^{ss}$ on an nontrivial open subset B_{i+1} of B_i (compare the proof of Lemma 2.12 in [H3]).

From this and the arguments in the proof of Corollary 2.13 of [H3] we conclude the statement of the proposition. **q.e.d.**

Let now M, N be closed negatively curved manifolds with isomorphic fundamental groups $\pi_1(M) = \pi_1(N) = \Gamma$. Recall that every isomorphism of $\pi_1(M)$ onto $\pi_1(N)$ induces a Γ-equivariant homeomorphism $f: \partial\tilde{M} \to \partial\tilde{N}$. The Lebesgue Liouville measure on T^1M (or T^1N) induces a Γ-invariant measure class λ_M (or λ_N) on $\partial\tilde{M}$ (or ∂N), and similarly the Bowen Margulis measure induces an invariant measure class μ_M (or μ_N). As an immediate consequence of Proposition 4.2 we obtain:

Corollary 4.3: *Assume that the Anosov splittings of T^1M and T^1N are of class C^1. Then every Γ-equivariant homeomorphism $f: \partial\tilde{M} \to \partial\tilde{N}$ which satisfies $f\{\lambda_M, \mu_M\} \cap \{\lambda_N, \mu_N\} \neq \emptyset$ is a C^1-diffeomorphism.*

Proof: Assume for example that $f\lambda_M = \lambda_N$. Then f^4 maps the Lebesgue Liouville cross ratio on $\partial\tilde{M}$ to the Lebesgue Liouville cross ratio on $\partial\tilde{N}$ up to a constant factor. Since the C^1-structures on $\partial\tilde{M}$ and $\partial\tilde{N}$ are determined by coordinate functions constructed out of the cross ratios (compare the proof of Proposition 4.2) we conclude that f is a C^1-diffeomorphism. **q.e.d.**

Recall that an *orbit-equivalence* of the geodesic flows on T^1M and T^1N is a homeomorphism $\Lambda: T^1M \to T^1N$ which maps every orbit of the geodesic flow on T^1M order preserving to an orbit of the geodesic flow on T^1N. Thus for every $v \in T^1M$ there is a homeomorphism $\alpha_v: \mathbf{R} \to \mathbf{R}$ such that $\Lambda(\Phi^t v) = \Phi^{\alpha_v(t)}\Lambda(v)$ for all $t \in \mathbf{R}$. If M and N have isomorphic fundamental groups, then their geodesic flows are orbit equivalent, moreover the orbit equivalence can be chosen to be smooth along the flow lines of the geodesic flow.

Corollary 4.4: *Under the hypothesis of Corollary 3.3 there is an orbit equivalence $\Lambda: T^1M \to T^1N$ of the geodesic flows which is a C^1-diffeomorphism.*

Proof: Assume that the hypothesis of Corollary 3.3 is satisfied; then $f\lambda_M = \lambda_N$ (after normalization). Let λ_M^{su} (or λ_N^{su}) be the family of Lebesgue measures on strong unstable manifolds which is induced from the lift of the Riemannian metric on M (or N). For $v \in T^1\tilde{M}$ let $\Lambda(v)$ be the unique vector in $T^1\tilde{N}$ such that $f(\pi(v)) = \pi(\Lambda(v)), f(\pi(-v)) = \pi(-\Lambda(v))$ and such that the Jacobian of f at $\pi(v)$ with respect to the projections of $\lambda_M^{su} \mid_{W^{su}(v)}$ and $\lambda_N^{su} \mid_{W^{su}(\Lambda(v))}$ equals 1. Then $v \to \Lambda(v)$ is a Γ-equivariant map which projects to an orbit equivalence of the geodesic flows on T^1M and T^1N.

We claim that Λ is in fact a C^1-diffeomorphism. To see this, recall that by our assumption the function trU on T^1M and T^1N is of class C^1 and hence for every $v \in T^1\tilde{M}$ the set

$$\bar{W}^{ss}(v) = \{w \in W^s(v) | d(\pi\lambda_M^{su} \mid_{W^{su}(v)})/d(\pi\lambda_M^{su}|_{W^{su}(w)})(\pi v) = 1\}$$

is a C^1-submanifold of $W^s(v)$. The restriction of $\pi \circ \mathcal{F}$ to $\bar{W}^{ss}(v)$ is a C^1-diffeomorphism of $\bar{W}^{ss}(v)$ onto $\partial\tilde{M} - \pi(v)$. But $\Lambda \mid_{\bar{W}^{ss}(v)} = (\pi \circ \mathcal{F} \mid_{\bar{W}^{ss}(\Lambda v)})^{-1} \circ f \circ (\pi \circ \mathcal{F} \mid_{\bar{W}^{ss}(v)})$ and hence Λ maps every stable manifold in T^1M C^1-diffeomorphically onto a stable manifold in T^1N. Similarly we also see that Λ maps every unstable manifold in T^1M C^1-diffeomorphically onto an unstable manifold in T^1N. In short, Λ is a C^1-diffeomorphism. **q.e.d.**

Recall that the geodesic flows on T^1M and T^1N are *homothetic* if there is a homeomorphism $\Lambda: T^1M \to T^1N$ and a number $a > 0$ such that $\Lambda(\Phi^t v) = \Phi^{at}(\Lambda v)$ for all $v \in T^1M$ and all $t \in \mathbf{R}$. In other words, the metric on N can be rescaled in such a way that after this rescaling the geodesic flows on T^1M and T^1N are time preserving conjugate.

Next we observe:

Lemma 4.5: *Let M, N be compact negatively curved manifolds. Assume that the Anosov splitting of T^1M and T^1N is of class C^1. If the geodesic flows on T^1M and T^1N are orbit equivalent with an orbit equivalence of class C^1, then they are homothetic.*

Proof: By our assumption, there is a $\Gamma = \pi_1(M) = \pi_1(N)$-equivariant diffeomorphism $f: \partial\tilde{M} \to \partial\tilde{N}$ of class C^1, and $f \times f$ is a C^1-diffeomorphism of the space $\mathcal{G}\tilde{M}$ of geodesics in $\tilde{M}$ onto the space $\mathcal{G}\tilde{N}$ of geodesics in $\tilde{N}$.

The symplectic form $d\omega_0$ on $T^1\tilde{N}$ is invariant under the geodesic flow and projects to a smooth symplectic form η_0 on $\mathcal{G}\tilde{N}$. Then $(f \times f)^*\eta_0$ is a continuous, $\pi_1(M)$-invariant non-degenerate 2-form on $\mathcal{G}\tilde{M}$ which can naturally be pulled back to a continuous, $\pi_1(M)$ invariant 2-form η on $T^1\tilde{M}$ which is invariant under the geodesic flow.

Now if $\Lambda: T^1M \to T^1N$ is an orbit equivalence of class C^1 induced as above by f, then naturally $\eta = \Lambda^* d\omega_0$ and hence the continuous form η is exact in the sense of distributions. By Theorem A of [H4], this means that necessarily $\eta = r d\omega$ for some $r \neq 0$, where $d\omega$ is the symplectic form on T^1M. The discussion in Section 3 of this paper then shows that $f \times f$ preserves the (usual) cross ratio on $\partial\tilde{M}$ and $\partial\tilde{N}$ up to a constant factor and hence the geodesic flows on T^1M and T^1N are homothetic. **q.e.d.**

As a corollary, we obtain Theorem C from the introduction:

Corollary 4.6: *Let M, N be closed negatively curved manifolds with isomorphic fundamental groups and Anosov splitting of class C^1. Then the natural boundary map $\partial\tilde{M} \to \partial\tilde{N}$ preserves the Lebesgue measure classes on $\partial\tilde{M}$ and $\partial\tilde{N}$ if and only if the geodesics flows on T^1M and T^1N are homothetic.*

Let L be the set of length cocycles of smooth metrics of negative curvature on M with Anosov splitting of class C^1. By Corollary 4.6 there is an injective map β of L into the space of geodesic currents $\mathcal{C}M$ of M which associates to a length cocycle α of a metric g the Lebesgue Liouville current of g. For every $\ell \in L$ and every geodesic current γ on M we then can define the *intersection* between γ and $\beta(\ell)$ by $i(\gamma, \beta(\ell)) = \int \ell d\gamma$.

Notice that this definition is an extension of the definition in the 2-dimensional case, however i is clearly not symmetric, i.e. in general we have $i(\beta(\ell), \beta(\ell')) \neq i(\beta(\ell'), \beta(\ell))$ for $\ell, \ell' \in L$.

We conjecture that this intersection can be extended to a continuous function on the product of the space of currents on M with the subspace of Gibbs currents, in particular Corollary 4.6 should hold without any regularity assumptions.

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